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CONTENTS

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MATRICES IN THE MARKET PLACE

F. D. PARKER, SUNY at Buffalo

Introduction. One often hears the story of the young American in a foreign bank who borrows ten dollars, exchanges them for francs, the francs for rupees, the rupees for pounds sterling, and the pounds sterling for dollars, all of which allows him to return the loan of ten dollars and pocket a profit. True or not, it would seem that such transactions are susceptible to mathematical treatment if an appropriate model is available. Such a model might be expected to find discrepancies in the foreign exchange system or any similar system. Since such systems are subjected to frequent, almost continuous, fluctuations (as well as incomplete information) it would seem that a shrewd manipulator could take advantage of such discrepancies to his financial gain.

As it happens, there is such a model, a model which has some nontrivial properties. The same model occurs in one aspect of group theory.

Equitable matrices. We begin by considering a system of n sets. Each set contains elements to which a value can be assigned (each element of a set has the same value). Each element of set S_i can be exchanged for a_{ij} elements of set S_j. Now the system can be described by a square matrix of order n whose entry in row i, column j is a_{ij}. To keep an application in mind, the elements of a given set can be dollars, bushels of wheat, shares of stock, one day's labor, etc. For such a system to be equitable, i.e., that there is no financial advantage to be gained by bartering, there must be certain conditions on the matrix.

1. $a_{ij}>0$ for all i, j.
2. $a_{ii}=1$ for all i.
3. $a_{ij}a_{ji}=1$ for all i, j.
4. $a_{ij}a_{jk}=a_{ik}$ for all i, j, k.

The fourth condition is the important one which demands that a transaction involving, for example, exchanging dollars for francs, then francs for rupees, yields as many rupees as the direct exchange of dollars for rupees.

In fact, these four conditions can be replaced by the following definition:

An *equitable matrix* is a square matrix of order n with positive entries such that $a_{ij}a_{jk}=a_{ik}$ for all i, j, k.

Equitable matrices have some interesting properties, some of which are described by the following theorems.

THEOREM I. *If M is an equitable matrix of order n, then $M^2=nM$.*

Proof. The element of M^2 in row i column j is given by

$$\sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} a_{ij} = na_{ij}.$$

THEOREM II. *If S is a square matrix of order n consisting entirely of unity elements, then S is an equitable matrix, and any equitable matrix (of order n) is similar to S.*

Proof. It is immediate that S is an equitable matrix. To show that M is similar to S, we can actually find the matrix P such that $P^{-1}MP=S$. Let $P=\text{diag}\ (1, a_{21}, a_{31}, \cdots a_{n1})$, then $P^{-1}=\text{diag}\ (1, a_{12}, a_{13}, \cdots, a_{1n})$. Then the element in the ij position of MP becomes $a_{ij}a_{j1}=a_{i1}$. Premultiplication by P^{-1} now provides in the ij position of $P^{-1}MP$ the element $a_{1i}a_{i1}=a_{11}=1$.

THEOREM III. *All equitable matrices of order n are similar, and their characteristic values are $(n, 0, 0, \cdots, 0)$.*

Proof. Since any equitable matrix is similar to S, they are all similar to each other. The characteristic values of S are easily shown to be $(n, 0, 0, \cdots, 0)$.

THEOREM IV. *Equitable matrices form a commutative group under Hadamard multiplication.* (The Hadamard product of $[a_{ij}]$ and $[b_{ij}]$ is $[a_{ij}b_{ij}]$.)

Proof. Consider two equitable matrices A and B. The entry in row i, column j of the Hadamard product is $a_{ij}b_{ij}$, and the entry in row j, column k is $a_{jk}b_{jk}$. But $a_{ij}b_{ij}a_{jk}b_{jk}=a_{ik}b_{ik}$, which is the entry in row i, column k. Hence equitable matrices are closed with respect to Hadamard multiplication. Associativity and commutativity are immediate, S serves as the identity and the transpose of a given equitable matrix is its inverse.

The group itself does not appear to be very interesting, being isomorphic to the direct product of $n-1$ groups, each group being the multiplicative group of positive real numbers. This is proved by realizing that only $n-1$ entries of an equitable matrix can be independent. To put it another way, as soon as the exchange rate is known for a unit of a given class in terms of a unit of each of the other classes (a row of the matrix), then the matrix is uniquely determined.

THEOREM V. *A matrix which diagonalizes an equitable matrix M is given by*

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ a_{21} & a_{21} & 0 & 0 & \cdots & 0 & 0 \\ a_{31} & -a_{31} & a_{31} & 0 & \cdots & 0 & 0 \\ a_{41} & 0 & -a_{41} & a_{41} & \cdots & 0 & 0 \\ \vdots & & & & & \vdots & \vdots \\ a_{n1} & 0 & 0 & 0 & \cdots & -a_{n1} & a_{n1} \end{bmatrix}.$$

Proof. Direct multiplication yields $MR=R\Lambda$, where $\Lambda=\text{diag}\ (n, 0, 0, \cdots, 0)$.

Inequitable matrices. What about inequitable matrices? How can they be discovered and exploited? Consider an ideal situation in which only one exchange rate is altered from an equitable situation. If, for example, the rate of exchange of a unit of commodity i for a unit of commodity j changes from a_{ij} to $a_{ij}+h$, then the matrix M^2-nM no longer is identically zero. The change occurs in row i, and in column j, and then M^2-nM is

$$\begin{bmatrix} & & ha_{1j} & & \\ & 0 & ha_{2j} & 0 & \\ & & \vdots & & \\ ha_{i1} & ha_{i2}\cdots & 2h & \cdots & ha_{in} \\ & & \vdots & & \\ & 0 & ha_{nj} & 0 & \end{bmatrix}.$$

An individual who owns k units of any commodity p can exchange them for ka_{pi} units of commodity i, then $ka_{pi}(a_{ij}+h)$ units of commodity j, and finally $ka_{pi}(a_{ij}+h)(a_{jp})$ units of his original commodity, thus realizing a profit of kha_{ji} units of commodity p. If h is negative, the transactions are made in the reverse order, yielding a profit of $-kh/(a_{ij}+h)$ units of commodity p.

Of course, this is an extremely unsophisticated model, but it is conceivable that an individual with reliable information on a wide market, and with high-speed computing facilities for keeping an almost continuous record of the matrix M^2-nM might be able to take advantage of a market discrepancy before the usual forces could react to restore equilibrium. The model does show, moreover, that a profit can be made by an individual owning any commodity, and that only three transactions are necessary. Thus we reach the conclusion that some of the transactions carried out by the "young American" were superfluous.

So far we have disregarded the usual fees which are associated with such transactions (commissions, taxes, brokers' fees, etc.). Suppose that such fees are directly proportional to the amount of the transaction, and that one unit of commodity p no longer is worth a_{pi} units of commodity i, but rather ra_{pi} units of commodity $i(r<1)$. If r is too small, then it is possible that no transaction can be profitable; in practice, r has a damping effect on the trading activity of a market. Again considering the case in which only one exchange rate is altered from an equitable rate a_{ij} to $a_{ij}+h$, we find now that the conversion of one unit of commodity p to commodity i to commodity j to commodity p now produces

$$r^3a_{pi}(a_{ij}+h)a_{jp}$$

units of commodity p. Unless h is large enough, there will be no profit. The condition for a profitable transaction is $hr^3>a_{ij}(1-r^3)$. If h is negative, the transactions are carried out in the opposite direction, and one unit of commodity p now becomes

$$r^3a_{pj}\frac{1}{a_{ij}+h}a_{ip}$$

units of the same commodity, and the condition for a profitable transaction is $-h>(1-r^3)a_{ij}$.

An application to group theory. In abstract algebra we frequently encounter a finite set G with a binary operation. If for any two elements a and b of G there is a unique solution to the equation $ax=b$ and a unique solution to the equation

$ya=b$, then G is a quasigroup. If at the same time there is a distinguished element e such that $ae=ea=a$ for all a in G, then G is a loop. If, in addition, the associative law $a(bc)=(ab)c$ holds for all a, b, c in G, then G is a group. Since groups have been extensively studied and possess much stronger theorems than do loops, it is often desirable to determine whether a given loop is a group. If the multiplication table of the loop is arranged so that the distinguished element e lies on the main diagonal (the normal form) and the table is considered as a matrix, then Zassenhaus [1] has shown that the associative law is equivalent to the condition $a_{ij}a_{jk}=a_{ik}$. This is precisely the requirement for an equitable matrix. Therefore, a necessary and sufficient condition that a loop be a group is that $P^{-1}MP=S$, where M is the Cayley table in normal form, P^{-1} and P are the diagonal matrices described in Theorem II, S is composed of entries all of which are the distinguished element e, and the operations are carried out in the binary operation of the system.

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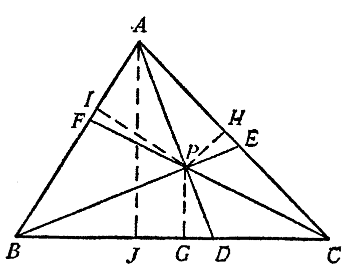
1. H. J. Zassenhaus, The theory of groups, Chelsea, New York, 1949.
2. F. D. Parker, When is a loop a group?, Amer. Math Monthly, 72 (1965) No. 7.

ON PEDAL RATIOS

D. MOODY BAILEY, Princeton, West Virginia

Through P, a point in the plane of triangle ABC, rays AP, BP, CP are constructed to meet sides BC, CA, AB at respective points D, E, F. From P perpendiculars are dropped to the sides of the triangle thereby determining points G, H, I. Triangles DEF and GHI are usually called the cevian and pedal triangles of point P with respect to triangle ABC. Sides BC, CA, AB will be represented by a, b, c as an effort is made to express pedal ratios BG/GC, CH/HA, AI/IB in terms of cevian ratios BD/DC, CE/EA, AF/FB and the sides of triangle ABC.

A beginning is made by considering the well-known equality $AP/PD=AE/EC+AF/FB$ [1, Theorem 1]. Suppose that positive unity be added to both members of this equation. Then $AP/PD+1=AE/EC+AF/FB+1$, or



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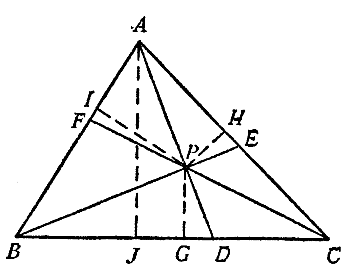

$AD/PD = AE/EC + AF/FB + 1$. Let AJ be the altitude from vertex A to side BC. As triangles AJD and PGD are similar right triangles, it is evident that $JD/GD = AD/PD = AE/EC + AF/FB + 1$. Now $JD/GD = (BD - BJ)/(BD - BG)$, from which $(BD - BJ)/(BD - BG) = AE/EC + AF/FB + 1$. Solving this equation for BG, it is found that

$$BG = \frac{BD\left(\dfrac{AE}{EC} + \dfrac{AF}{FB}\right) + BJ}{\dfrac{AE}{EC} + \dfrac{AF}{FB} + 1}. \tag{1}$$

From Ceva's theorem it is known that $(BD/DC) \cdot (CE/EA) \cdot (AF/FB) = 1$, or $AF/FB = (AE/EC) \cdot (CD/DB)$. This value of AF/FB may be substituted in the expression $BD(AE/EC + AF/FB)$ which occurs in the numerator of (1). Hence

$$\begin{aligned} BD\left(\frac{AE}{EC} + \frac{AF}{FB}\right) &= BD\left(\frac{AE}{EC} + \frac{AE}{EC}\cdot\frac{CD}{DB}\right) = BD\left(1 + \frac{CD}{DB}\right)\frac{AE}{EC} \\ &= BD\left(1 + \frac{DC}{BD}\right)\frac{AE}{EC} = a\cdot\frac{AE}{EC}. \end{aligned}$$

Then (1) becomes

$$BG = \frac{a\cdot\dfrac{AE}{EC} + BJ}{\dfrac{AE}{EC} + \dfrac{AF}{FB} + 1}.$$

It is immediately apparent that

$$GC = \frac{a\cdot\dfrac{AF}{FB} + JC}{\dfrac{AE}{EC} + \dfrac{AF}{FB} + 1},$$

since the relationship $BG + GC = a$ is then satisfied. Using these values of BG and GC, we obtain

$$\frac{BG}{GC} = \frac{a\cdot\dfrac{AE}{EC} + BJ}{a\cdot\dfrac{AF}{FB} + JC}.$$

Since AJ is an altitude of triangle ABC, it follows that $BJ = c \cos B$ and $JC = b \cos C$. The well-known triangle formulas $b^2 = a^2 + c^2 - 2ac \cos B$ and

$c^2=a^2+b^2-2ab\cos C$ may be used to obtain $c\cos B=(a^2+c^2-b^2)/2a$ and $b\cos C=(a^2+b^2-c^2)/2a$. We are thus able to write

$$\frac{BG}{GC}=\frac{a\cdot\dfrac{AE}{EC}+BJ}{a\cdot\dfrac{AF}{FB}+JC}=\frac{a\cdot\dfrac{AE}{EC}+\dfrac{a^2+c^2-b^2}{2a}}{a\cdot\dfrac{AF}{FB}+\dfrac{a^2+b^2-c^2}{2a}}=\frac{2a^2\dfrac{AE}{EC}+a^2+c^2-b^2}{2a^2\dfrac{AF}{FB}+a^2+b^2-c^2}.$$

In a similar fashion values for CH/HA and AI/IB may be obtained.

THEOREM. *P is any point in the plane of triangle ABC, with DEF its cevian triangle. Perpendiculars from point P meet sides BC, CA, AB at respective points G, H, I so that*

$$\frac{BG}{GC}=\frac{2a^2(AE/EC)+a^2+c^2-b^2}{2a^2(AF/FB)+a^2+b^2-c^2},\qquad \frac{CH}{HA}=\frac{2b^2(BF/FA)+a^2+b^2-c^2}{2b^2(BD/DC)+b^2+c^2-a^2},$$
$$\frac{AI}{IB}=\frac{2c^2(CD/DB)+b^2+c^2-a^2}{2c^2(CE/EA)+a^2+c^2-b^2}.$$

As an example of the use of the theorem allow P to be the centroid of triangle ABC. Then $(BD/DC)=(CE/EA)=(AF/FB)=1$ and

$$\frac{BG}{GC}=\frac{3a^2+c^2-b^2}{3a^2+b^2-c^2},\qquad \frac{CH}{HA}=\frac{3b^2+a^2-c^2}{3b^2+c^2-a^2},\qquad \frac{AI}{IB}=\frac{3c^2+b^2-a^2}{3c^2+a^2-b^2}.$$

When P is the incenter $BD/DC=c/b$, $CE/EA=a/c$, $AF/FB=b/a$ and

$$\frac{BG}{GC}=\frac{a+c-b}{a+b-c},\qquad \frac{CH}{HA}=\frac{a+b-c}{b+c-a},\qquad \frac{AI}{IB}=\frac{b+c-a}{a+c-b}.$$

In this instance the converse of Ceva's theorem shows that rays AG, BH, CI are concurrent at a point known as the Gergonne point of the triangle. The reader will find it worth while to compute BG/GC, CH/HA, AI/IB for the symmedian point, circumcenter, and Brocard points of triangle ABC.

The author has found the theorem to be of value when dealing with properties of the pedal triangles of pairs of isogonal conjugates in a given triangle. It may be used to determine the points at which the radical axis of circumcircle ABC and pedal circle GHI meet the sides of triangle ABC. Many facts concerning the Simson lines of points on circumcircle ABC may be obtained through its use.

In using the theorem it should be remembered that points D and G lie between vertices B and C when ratios BD/DC and BG/GC are positive. The points lie on BC extended when the corresponding ratios are negative. Similar comments apply to the other ratios involved in the results given.

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A NOTE ON EXACT DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

PHILIP FUNG, Fenn College

A "short-cut" method usually employed by students in solving an exact differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{1}$$

involves the representation of its integral by the sum of all distinct terms in the integrals $\int M(x, y)dx$ and $\int N(x, y)dy$. This method, though applicable to many cases, fails to be valid in some, as illustrated in example 2. It is the purpose of this note to justify the feasibility of such a method.

DEFINITION. *Two functions f and g are equivalent (written $f \simeq g$) if they differ by an additive constant only.*

We write $U_1(x, y) = \int M(x, y)dx = \sum_i h_i(x, y) + \sum_j m_j(x, y)$ and $U_2(x, y) = \int N(x, y)dy = \sum_i h_i(x, y) + \sum_k n_k(x, y)$ where $m_j \not\simeq n_k \ \forall j, k$. Let $U(x, y) = \sum h_i(x, y) + \sum m_j(x, y) + \sum n_k(x, y)$. Then $U = U_1 + \sum n_k = U_2 + \sum m_j$.

THEOREM. *$U = C$ (C a constant) is the integral of (1) if and only if $m_j(x, y) = f_j(x)$ and $n_k(x, y) = g_k(y) \ \forall j, k$.*

Proof. $U = C$ is the integral of (1) iff

$$\frac{\partial U}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial U}{\partial y} = N(x, y)$$

i.e.,

$$\frac{\partial(U_1 + \sum n_k)}{\partial x} = M \quad \text{and} \quad \frac{\partial(U_2 + \sum m_j)}{\partial y} = N$$

i.e.,

$$\frac{\partial U_1}{\partial x} + \sum \frac{\partial n_k}{\partial x} = M \quad \text{and} \quad \frac{\partial U_2}{\partial y} + \sum \frac{\partial m_j}{\partial y} = N$$

i.e.,

$$\sum \frac{\partial n_k}{\partial x} = 0 \quad \text{and} \quad \sum \frac{\partial m_j}{\partial y} = 0 \quad \left(\text{since } \frac{\partial U_1}{\partial x} = M \quad \text{and} \quad \frac{\partial U_2}{\partial y} = N\right)$$

iff

$$n_k = g_k(y) \quad \text{and} \quad m_j = f_j(x) \ \forall j, k.$$

As illustrations, we consider the following examples.

Example 1. Solve: $2(xy+1)dx + (x^2+1)dy = 0$.

Here

$$U_1 = \int 2(xy + 1)dx = x^2y + 2x,$$

and

$$U_2 = \int (x^2 + 1)dy = x^2y + y.$$

Thus the solution is $x^2y+2x+y=C$.

Example 2. Solve:

$$\left[y+\frac{y}{(x+y)^2}\right]dx+\left[x+2y-\frac{x}{(x+y)^2}\right]dy=0.$$

Now $U_1=xy-y/(x+y)$ and $U_2=xy+y^2+x/(x+y)$. From the theorem, we know that $xy+y^2-y/(x+y)+x/(x+y)=C$ cannot be a solution. Since $x/(x+y) \simeq -y/(x+y)$, the solution is $xy+y^2+x/(x+y)=C$.

A NEW APPROACH TO INTEGRATION FOR FUNCTIONS OF A COMPLEX VARIABLE

DONALD H. TRAHAN, University of Pittsburgh

1. Introduction. The aim of this paper is to present a new approach to complex integration which the author feels is valuable for its simplicity and within easy grasp of the first year student of complex variables. In certain respects this approach to complex integration is more general than is usually found in textbooks. Some of the theorems that are derived in this paper are known, but they have appeared in isolation, usually in older journals and as exercises in textbooks. By means of the approach used in this paper, improvements were made in stating and proving theorems. Several ideas and theorems appear to be new, and the approach is quite interesting in itself.

The first theorem of this paper expresses the contour integral of a complex function as an area integral. The converse problem is one of the main problems of this paper, and in sections 3 and 4 several theorems are given which express an area integral as a contour integral. In sections 5, 6, and 7 other related topics are considered. Section 5 includes a generalization of the fact that e^x is its own derivative. In section 6 a relationship between the operator δf, introduced in this paper, and the Laplacian operator is established and applied. Section 7 includes generalizations of several basic theorems of complex integration theory.

2. Notation. Some innovations in notation are used in this paper in order to facilitate manipulations. In order to handle the factor i easily, we use $dA =dx(idy)$. In general $f=U(x, y)+iV(x, y)$ where U, V are real-valued functions. The customary symbols for the directional derivatives are used: $f_x' = U_x +iV_x$ and $f_y' = V_y-iU_y$. The author wishes to stress the importance of the difference of the directional derivatives and therefore uses the operator notation: $\delta f=f_x' - f_y'$, $\delta^k f=(\delta^{k-1}f)_x' - (\delta^{k-1}f)_y'$ where k is any positive integer and $\delta^0 f=f$. Thus in this notation if f is regular on a domain D then $\delta f=0$. It is easy to show that the operator δ is a linear operator, and in general that δ^k is a linear operator where k is any nonnegative integer. It is true that $\delta f=2(\partial f/\partial \bar{z})$, where differentiation is with respect to the complex conjugate of z. (The last concept is used in several books, including [1] and [4].) However, the author believes that the

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3. Preliminary theorems and corollaries. The first two theorems are known. Theorem 1 is usually considered as a complex form of Green's theorem. Theorem 2 is given in a less compact form by R. Nevanlinna [**5**].

THEOREM 1. *If the closed region A is encompassed by a simple, rectifiable path C and f_x', f_y' are continuous on A, then*

$$\int_C f\,dz = \iint_A \delta f\,dA.$$

Proof. It follows that f is continuous on A; consider

$$\int_C f\,dz = \int_C (Udx - Vdy) + i\int_C (Vdx + Udy).$$

Applying Green's theorem to the last equation gives

$$\int_C f\,dz = -\iint_A (V_x + U_y)dxdy + i\iint_A (U_x - V_y)dxdy = \iint_A \delta f\,dA.$$

THEOREM 2. *If the closed region A is encompassed by a simple, rectifiable path C and f_x', f_y', g_x', g_y' are continuous on A, then*

$$\int_C fg\,dz = \iint_A (f\delta g + g\delta f)dA.$$

Proof. This follows from Theorem 1, since $(fg)_x' = fg_x' + gf_x'$ and $(fg)_y' = fg_y' + gf_y'$.

Using Theorem 2 the reader can write formulas for the contour integral of f/g and for f^n. The following corollaries seem more interesting, and Corollaries 2 and 3 give special solutions to the problem of expressing the area integral of a function as a contour integral.

COROLLARY 1. *If the closed region A is encompassed by a simple, rectifiable path C, f_x', f_y' are continuous on A, and g is regular on A, then*

$$\int_C gf\,dz = \iint_A g\delta f\,dA.$$

Proof. This corollary follows by Theorem 2 since $\delta g = 0$.

COROLLARY 2. *If the closed region A is encompassed by a simple, rectifiable path C and f is regular on A, then*

$$\iint_A f\,dA = \int_C xf\,dz = \int_C -iyf\,dz.$$

Proof. In Theorem 2, let $g=x$; then $\delta g=1$, $\delta f=0$ and this gives the first result. The second result follows from the first, since zf is also regular on A.

COROLLARY 3. *If the closed region A is encompassed by a simple, rectifiable path C and $f=f(x)$ is continuous on A, then*

$$\iint_A f(x)\,dA = \int_C \left[\int f(x)\,dx\right] dz.$$

Proof. This corollary follows by applying Theorem 1 to the contour integral.

Of course, there is a similar formula if f is a function of y (just replace dx of Corollary 3 by $-idy$); this is also true of the results in sections 4 and 5—one can obtain another formula in each case by replacing x by $(-iy)$.

COROLLARY 4. *If the closed region A is encompassed by a simple, rectifiable path C and f_x' is continuous on A, and, in addition, if $U_y=-V_x$, then $\int_C U dz=\iint_A \bar{f}_x' dA$ and if $U_x=V_y$, then $\int_C iV dz=-\iint_A \bar{f}_x' dA$.*

Proof. These results follow by Theorem 1, since for example $\delta U=U_x+iU_y=\bar{f}_x'$.

Of course, there is a similar corollary with f_y' replacing f_x'. The next corollary follows from Corollary 4. It also follows directly from Theorem 1, since $\delta(\bar{f})=2(\bar{f}')$.

COROLLARY 5. *If the closed region A is encompassed by a simple, rectifiable path C and f is regular on A, then*

$$\int_C \bar{f}\,dz = 2\iint_A \bar{f}'\,dA.$$

Corollary 5 and several other interesting results that follow from Theorem 1 are given by R. Nevanlinna ([**5**] pp. 126–127). The main reason for giving Corollary 5 in this paper is that it will be used in section 6.

4. The main theorems for conversion of area integrals.

LEMMA. *If the closed region A is encompassed by a simple, rectifiable path C and $(\delta^k f)_x'$, $(\delta^k f)_y'$ are continuous on A, then for every nonnegative integer k*

$$\int_C \frac{x^{k+1}}{(k+1)!}\delta^k f\,dz = \iint_A \left[\frac{x^{k+1}}{(k+1)!}\delta^{k+1}f + \frac{x^k}{k!}\delta^k f\right] dA.$$

Proof. In Theorem 2 let $(k+1)!\,g=x^{k+1}$ and replace f by $\delta^k f$.

THEOREM 3. *If the closed region A is encompassed by a simple, rectifiable path C and $(\delta^k f)_x'$, $(\delta^k f)_y'$ $(k=0, 1, 2, \cdots, n-1)$ are continuous on A, then*

$$\iint_A f\,dA = \int_C \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k!}\delta^{k-1}f\,dz + \iint_A \frac{(-x)^n}{n!}\delta^n f\,dA.$$

Proof. By the Lemma

$$\int_C xf\,dz = \iint_A (x\delta f + f)\,dA$$

$$-\int_C \frac{x^2}{2!}\delta f\,dz = -\iint_A \left[\frac{x^2}{2!}\delta^2 f + x\delta f\right] dA$$

. . . .

$$(-1)^{n-1}\int_C \frac{x^n}{n!}\delta^{n-1}f\,dz = (-1)^{n-1}\iint_A \left[\frac{x^n}{n!}\delta^n f + \frac{x^{n-1}}{(n-1)!}\delta^{n-1}f\right] dA.$$

Adding these equations gives the result.

COROLLARY 6. *If the closed region A is encompassed by a simple, rectifiable path C and $\delta^{n-1}f$ is regular on A, then*

$$\iint_A f\,dA = \int_C \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k!}\delta^{k-1}f\,dz\cdot$$

Corollary 6 is a generalization of Corollary 2.

THEOREM 4. *If the closed region A is encompassed by a simple, rectifiable path C, $(\delta^k f)'_x$, $(\delta^k f)'_y$ $(k=0, 1, 2, \cdots)$ are continuous on A, and the series*

$$\sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^k}{k!}\delta^{k-1}f$$

converges uniformly on A, then

$$\iint_A f\,dA = \int_C \left[\sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^k}{k!}\delta^{k-1}f\right] dz.$$

Proof. The series represents a continuous function on A and hence it has a contour integral. The result can be obtained from Theorem 3, since the last integral given in the equation of Theorem 3 converges to zero as n increases.

THEOREM 5. *If the closed region A is encompassed by a simple, rectifiable path C, $(\delta^k f)'_x$, $(\delta^k f)'_y$ $(k=0, 1, 2, \cdots, n-1)$ are continuous on A, and $\delta^n f$ is a function of x on A, then*

$$\iint_A f\,dA = \int_C \left[\sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k!}\delta^{k-1}f\right] dz + \int_C \left[\int \frac{(-x)^n}{n!}\delta^n f\,dx\right] dz.$$

Proof. This theorem follows by Corollary 3 and Theorem 3.

5. Two classes of functions: $\delta^n f = g$ where g is regular and $\delta f = f$. In this section two classes of functions are considered: functions f such that $\delta^n f = g$ where g is a known regular function and functions f such that $\delta f = f$. (Functions such that $\delta^n f = 0$ are discussed in a paper by A. Kriszten [**3**].)

THEOREM 6. *If $(\delta^k f)'_x$, $(\delta^k f)'_y$ $(k=0, 1, 2, \cdots, n-1)$ are continuous on a domain D and g is regular on D, then $\delta^n f = g$ on D if and only if*

$$f = g\frac{x^n}{n!} + \sum_{k=0}^{n-1} h_k \frac{x^k}{k!},$$

where the h_k are regular functions on D.

Proof. By Theorem 1 and Corollary 2, for any closed path C in D,

$$\int_C \delta^{n-1} f\, dz = \iint_A g\, dA = \int_C gx\, dz.$$

Therefore, since C is arbitrary, $\delta^{n-1}f - gx = h_{n-1}$ where h_{n-1} is regular on D. Similarly, by Theorem 1 and Corollaries 1, 2, 3 we have $\delta^{n-2}f = gx^2/2! + h_{n-1}x + h_{n-2}$. The result follows by induction. The converse is easily verified.

COROLLARY 7. *If $P_n(x, y)$ is a polynomial of degree n in two real variables, then*

$$P_n(x, y) = \sum_{k=0}^{n} h_k \frac{x^k}{k!},$$

where the h_k are regular functions.

Proof. This corollary follows from Theorem 6, since $\delta^{n+1}(P_n) = 0$.

THEOREM 7. *Let f'_x, f'_y be continuous on a domain D; then $\delta f = f$ on D if and only if $f = he^x$ where h is a regular function on D.*

Proof. By Theorems 1 and 4

$$\int_C f\, dz = \iint_A f\, dA = \int_C f(1 - e^{-x})dz,$$

so that the contour integral of fe^{-x} is zero. Since C is arbitrary it follows that fe^{-x} is a regular function on D. The converse is easily verified.

Theorem 7 is a generalization of the following very well-known property: if $f = f(x)$, then $f = e^x$ if and only if $Df = f$. By Theorem 1 this also means that with C arbitrary in D, $\iint_A he^x dA = \int_C he^x dz$.

6. The relationship between δf and the Laplacian operator. Many of the theorems and corollaries that have been established in this paper can be reinterpreted in terms of the operator δ, since in a domain D the problem of expressing an area integral of a function f as a contour integral is equivalent to finding a function g such that $\delta g = f$ on D or so that $g = \delta^{-1} f$. Let f be a known function on a domain D, and suppose it is necessary to find all functions of the form $\delta^{-1} f$ on D; then if g is a particular solution, all solutions are of the form $g + h$ where h is any regular function on D, since $\delta(g+h) = \delta g$. For example, Corollary 2 also means that if f is regular on a domain D then $\delta^{-1} f = xf$, and therefore all solutions are of the form $xf + h$ where h is any regular function on D.

The operator δf and the Laplacian operator are closely related. In fact, let $\alpha f = i\widetilde{\delta f}$ where $\widetilde{\delta f}$ is the complex conjugate of δf; then $\alpha f = (V_x + U_y) + i(U_x - V_y)$ and $\alpha^2 f = (U_{xx} + U_{yy}) + i(V_{xx} + V_{yy}) = \Delta f$, where Δ is the Laplacian operator.

This gives a new way of looking at the close relationship between functions which are regular and those which are harmonic. For f is regular implies $\alpha f = 0$; f is harmonic implies $\alpha^2 f = 0$; and f is bi-harmonic implies $\alpha^4 f = 0$. Therefore, it may be desirable to call harmonic functions, 2-regular functions, and in general to define f to be n-regular if f has continuous partial derivatives of the nth order and $\alpha^n f = 0$. In particular, it might be worthwhile to study 3-regular functions. It should be noted that if f is n-regular then $\alpha^{n-k} f$ is k-regular.

In applied mathematics there are many occasions when it is necessary to solve a partial differential equation of the form $\Delta U = P$, where P is known and $U = U(x, y)$, $P = P(x, y)$ are real-valued functions of two real variables on a domain D. Next we consider the related problem of finding the inverse of the Laplacian operator Δ.

As given above, $\alpha f = i\widetilde{\delta f}$ and $\alpha^2 f = \Delta f$. Suppose $\alpha f = g$; then $i\widetilde{\delta f} = g$ and therefore $\widetilde{\delta f} = -ig$, $\delta f = i\tilde{g}$, $f = \delta^{-1}(i\tilde{g}) = i\delta^{-1}(\tilde{g})$. Therefore, $\alpha^{-1}(g) = i\delta^{-1}(\tilde{g})$ and the inverse, Δ^{-1}, of the Laplacian operator Δ is equal to α^{-2}, where α^{-2} means two successive applications of α^{-1}.

First, consider the problem of finding the most general function f such that $\Delta f = 0$ on a domain D. The first application of α^{-1} gives $\alpha^{-1}(0) = i\delta^{-1}(0) = ih_1 = h_2$ where h_1, h_2 are arbitrary regular functions on D. Applying α^{-1} again and using the relation indicated by Corollary 5, yields $\alpha^{-1}(h_2) = i\delta^{-1}(\tilde{h}_2) = (i/2)\tilde{h}_3 + f_2 = \tilde{f}_1 + f_2$ where h_3, f_1, f_2 are arbitrary regular functions on D, and h_3 is the indefinite integral of h_2 with respect to z. Therefore, $\Delta f = 0$ on D if and only if $f = \tilde{f}_1 + f_2$ where f_1, f_2 are regular functions on D.

Next consider the more general problem of solving $\Delta f = g$ on a domain D, where g is a known regular function on D. From the preceding example, it follows that it will be sufficient to find one solution. The following is a solution: $\alpha^{-1}(g) = (i/2)\tilde{h}$ where $h = \int_{z_0}^{z} g\,dz$ is the indefinite integral of g with respect to z. Applying α^{-1} again, yields $\alpha^{-1}[(i/2)\tilde{h}] = -\frac{1}{2}\delta^{-1}(h) = -(x/2)h$. Therefore, if g is regular on D, $-x/2\int_{z_0}^{z} g\,dz$ is a solution to $\Delta f = g$ on D, and any other solution can be obtained by adding $\tilde{f}_1 + f_2$ to $-x/2\int_{z_0}^{z} g\,dz$ where f_1, f_2 are regular functions on D. In addition, the result obtained by using only the upper limit of integration is also a solution, in fact, a somewhat simpler solution.

This last result can be used to give an especially simple method of solving $\Delta U = P(x, y)$ for $U = U(x, y)$ on a domain D, provided P is harmonic on D. For if P is harmonic one can determine $Q(x, y)$ such that $P + iQ$ is regular on D. By the above method one can solve $\Delta f = g$, and if $f_1 = U_1 + iV_1$ is one solution to $\Delta f = g$ then U_1 is a solution to $\Delta U = P$.

For example, let us solve $\Delta U = xy$. Then $P = xy$, $g = -(i/2)z^2$, $f_1 = (i/12)xz^3$, and $U_1 = [3x^2 - y^2]xy/12$ is one solution.

This type of problem is also considered by Z. Nehari ([4] pp. 36–40). The method given in Nehari's book involves using $\Delta f = 4(\partial^2 f)/(\partial z \partial \bar{z})$ and integrating first with respect to z and then with respect to the conjugate of z. In this way, one can solve $\Delta U = P$ even if P is not harmonic. However, there would be cases, especially when P is not a polynomial in x and y, when the method given here would be simpler.

7. Some generalizations of analytic function theory.

THEOREM 8. *If the closed region A is encompassed by a simple, rectifiable path C, $z_1 \notin A$ and f_x', f_y' are continuous on A, then, for every integer k,*

$$\int_C \frac{f(z)}{(z-z_1)^k}\,dz = \iint_A \frac{\delta f}{(z-z_1)^k}\,dA.$$

Proof. This theorem follows from Theorem 2, since $(z-z_1)^{-k}$ is regular on A.

THEOREM 9. *If f_x', f_y' are continuous on the closed, annular region R encompassed by the closed paths C and C_1 (C_1 interior to C) and $z_1 \notin R$, then, for every integer k,*

$$\int_C \frac{f(z)}{(z-z_1)^k}\,dz = \int_{C_1} \frac{f(z)}{(z-z_1)^k}\,dz + \iint_R \frac{\delta f}{(z-z_1)^k}\,dA.$$

Proof. Connect the paths C and C_1 by two line segments in the annular region. This decomposes R into two subregions and applying Theorem 8 to these regions gives the result.

THEOREM 10. *If the closed region A is encompassed by a simple, rectifiable path C, the closed regions A_k are encompassed by simple, rectifiable paths C_k ($k=1, 2, \cdots, n$), the C_k are in the interior of C, $A_k \cap A_j = \varnothing$ ($k \neq j$), and f_x', f_y' are continuous on the closed, annular region R, then*

$$\int_C f\,dz = \sum_{k=1}^{n} \int_{C_k} f\,dz + \iint_R \delta f\,dA.$$

Proof. By introducing line segments joining the paths we can divide the region R into two subregions. The result follows by applying Theorem 1 to these two subregions and adding the integrals.

DEFINITION (*of a residue*). *If C is a circle of radius r about z_1, then the residue of $f(z)$ at z_1 is given by*

$$R[f \mid z = z_1] = \operatorname*{Lim}_{r\to 0} \frac{1}{2\pi i} \int_C f\,dz.$$

THEOREM 11 (*a residue theorem*). *If the closed region A is encompassed by a simple, rectifiable path C and f_x', f_y' are continuous on A except for interior points z_k ($k=1, 2, \cdots, n$), then*

$$\int_C f\,dz - \iint_A \delta f\,dA = 2\pi i \sum_{k=1}^{n} R[f \mid z = z_k].$$

Proof. The result follows easily from Theorem 10.

ILLUSTRATION. The residue theorem gives a simple method for calculating some double integrals. For example, let $f=x(x^2+y^2)^{-1}$; then $\delta f=-z^2(x^2+y^2)^{-2}$. Let C_1 be a small circle of radius r and area A_1 with center at $z=0$; then the residue at $z=0$ can be found by using $r^{-2}\int_{C_1}x\,dz=iA_1r^{-2}=i\pi$, where the integral is evaluated by Corollary 2. Therefore, $\frac{1}{2}=R[f|z=0]$ and

$$\int_C x(x^2+y^2)^{-1}dz+\iint_A z^2(x^2+y^2)^{-2}dA=\pi i,$$

provided $z=0$ is an interior point of A. In particular, let $C=C_0+C_1$ where C_0 consists of that arc of the unit circle with center at the origin that goes from

$$z_0=\frac{-1-\sqrt{3}i}{2}\quad\text{to}\quad z_1=\frac{-1+\sqrt{3}i}{2}$$

and C_1 consists of the line segment joining z_1 to z_0. Then

$$\int_{C_0}f\,dz=\int_{C_0}x\,dz=\int_C x\,dz-\int_{C_1}x\,dz=\left[\frac{2\pi}{3}-\frac{\sqrt{3}}{4}\right]i,$$

where, by Corollary 2, $\int_C x\,dz=iA$ and the area A can easily be evaluated by introducing line segments joining z_0 to 0 and z_1 to 0. Since $\int_{C_1}f\,dz=2\pi i/3$ then

$$\int_C f\,dz=\left[\frac{4\pi}{3}-\frac{\sqrt{3}}{4}\right]i.$$

Therefore,

$$\iint_A z^2(x^2+y^2)^{-2}dA=\frac{1}{2}+\left[\frac{\sqrt{3}}{4}-\frac{\pi}{3}\right]i,$$

$$\iint_A (x^2-y^2)(x^2+y^2)^{-2}dxdy=\frac{\sqrt{3}}{4}-\frac{\pi}{3},$$

and

$$\iint_A xy(x^2+y^2)^{-2}dxdy=0.$$

THEOREM 12. *If* $\operatorname{Lim}_{z\to z_1}(z-z_1)f(z)=0$, *then* $0=R[f|z=z_1]$.

Proof. Given $\epsilon>0$ we can select a circle C about z_1 such that $|z-z_1|\;|f(z)|<\epsilon$ if $z\in C$; therefore

$$\left|\int_C f\,dz\right|<\epsilon r^{-1}\int_C|dz|=2\pi\epsilon.$$

THEOREM 13. *If the closed region A is encompassed by a simple, rectifiable path C, f'_x, f'_y are continuous on A except for a finite number of interior points z_k, and for each z_k* $\operatorname{Lim}_{z\to z_k}(z-z_k)f(z)=0$, *then*

$$\int_C f\,dz=\iint_A \delta f\,dA.$$

Theorem 13 follows from Theorems 11 and 12 and is a generalization of Theorem 1. This type of theorem for analytic functions is given by L. Ahlfors ([1] pp. 90–92), but the approach used here is entirely different.

The next theorem is a generalization of Cauchy's integral formula. It is given in a less compact form by E. R. Hedrick [2], but it is included here in an attempt to show the clarity and unity derived from using this approach to integration. First there is the following Lemma. (Actually, the Lemma is one step of the usual proof of Cauchy's integral formula as given in most textbooks.)

LEMMA. *If $f(z)$ is continuous on a domain D and $z_1 \in D$, then*

$$2\pi i f(z_1) = \operatorname*{Lim}_{r\to 0} \int_C \frac{f(z)}{z - z_1}\, dz,$$

where C is a circle of radius r about z.

Proof. As in the proof of Cauchy's integral formula, the result follows from

$$\int_C \frac{f(z)}{z - z_1}\, dz = 2\pi i f(z_1) + \int_C \frac{f(z) - f(z_1)}{z - z_1}\, dz.$$

THEOREM 14 (*A generalization of Cauchy's integral formula*). *If the closed region A is encompassed by a simple, rectifiable path C, $z_1 \in A_i$, and f_x', f_y' are continuous on A, then*

$$2\pi i f(z_1) = \int_C \frac{f(z)}{z - z_1}\, dz - \iint_A \frac{\delta f}{z - z_1}\, dA.$$

Proof. Let C_1 be a small circle of radius r about z_1 in A and let R be the annular region; then, by Theorem 9,

$$\int_C \frac{f(z)}{z - z_1}\, dz = \int_{C_1} \frac{f(z)}{z - z_1}\, dz + \iint_R \frac{\delta f}{z - z_1}\, dA.$$

The result follows by letting r approach zero and using the preceding Lemma.

COROLLARY 8. *If the closed region A is encompassed by a simple, rectifiable path C, f is regular on A, and $z_1 \in A_i$, then*

$$2\pi i \widetilde{f(z_1)} = \int_C \frac{\widetilde{f(z)}}{z - z_1}\, dz - 2 \iint_A \frac{\widetilde{f'(z)}}{z - z_1}\, dA.$$

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USE OF HYPERBOLIC SUBSTITUTION FOR CERTAIN TRIGONOMETRIC INTEGRALS

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In the "good old days," that is, the first half of this century, there was taught in at least one of the better engineering colleges in the United States, an ingenious method of integration of certain trigonometric expressions, which is now on the way to becoming a lost art. Not only is it ingenious, but it includes some very good basic mathematics.

As taught to this writer, the method was theoretically faulty, but it worked—that is, it gave results of integration problems which could be verified to be correct by the reverse process of differentiation. The mathematics teachers at the above-mentioned college at the time were engineers rather than mathematicians, and so did not concern themselves with what they considered to be mathematical technicalities; they were interested only in results.

The method was called by its author "imaginary substitution," since it involved complex numbers. A more accurate term would be hyperbolic substitution. It is applicable to integrals of the form

$$\int \cos^m x \sin^n x \, dx$$

where m and n are integers and $m+n$ is odd negative. For example,

	m	n	$m+n$
$\int \tan^2 x \sec x \, dx = \int \cos^{-3} x \sin^2 x \, dx$	-3	2	-1
$\int \csc^3 x \, dx = \int \sin^{-3} x \, dx$	0	-3	-3
$\int \dfrac{dx}{\cos x \sin^2 x}$	-1	-2	-3

These integrations done by any other method may be extremely tedious.

The method illustrated here is a mathematically correct version of the original method.

The substitution set used is the appropriate one of the following two:

Substitution set I: Let $\tan x = \sinh \theta$. Then $\sec^2 x \, dx = d(\tan x) = \cosh \theta \, d\theta$ and

$$\sec x = \sqrt{(1 + \tan^2 x)} = \sqrt{(1 + \sinh^2 \theta)} = \cosh \theta \qquad (-\pi/2 < x < \pi/2).$$

Substitution set II: Let $\cot x = \sinh \theta$. Then $-\csc^2 x \, dx = d(\cot x) = \cosh \theta \, d\theta$ and

$$\csc x = \sqrt{(1 + \cot^2 x)} = \sqrt{(1 + \sinh^2 \theta)} = \cosh \theta \qquad (0 < x < \pi/2).$$

Both of these transformations are 1-1 and monotone, and, therefore, acceptable.

In using this method, it will need to be recalled that $e^\theta=\cosh\theta+\sinh\theta$ from which it follows that $\theta=\ln(\cosh\theta+\sinh\theta)$.

How these substitutions are used is shown in the following examples.

Example 1. $\int\sec x\,dx=\int(\cos x)^{-1}dx$ $\qquad(m=-1,\ n=0,\ m+n=-1)$.

$$\int\sec x\,dx=\int\frac{\sec^2 x\,dx}{\sec x}=\int\frac{d(\tan x)}{\sec x}.$$

Using substitution set I, we obtain

$$\int\frac{\cosh\theta\,d\theta}{\cosh\theta}=\int d\theta=\theta=\ln(\cosh\theta+\sinh\theta)=\ln(\sec x+\tan x).$$

Example 2. $\int\csc^3 x\,dx=\int\sin^{-3}x\,dx$ $\qquad(m=0,\ n=-3,\ m+n=-3)$,

$$\int\csc^3 x\,dx=-\int\csc x(-\csc^2 x\,dx)=-\int\csc x\,d(\cot x).$$

Using substitution set II, we obtain

$$-\int(\cosh\theta)(\cosh\theta\,d\theta)=-\int\cosh^2\theta\,d\theta=-\frac{1}{2}\int(1+\cosh 2\theta)d\theta$$

$$=-\frac{1}{2}\int(d\theta+\cosh 2\theta\,d\theta)=-\frac{1}{2}\theta-\frac{1}{4}\sinh 2\theta$$

$$=-\frac{1}{2}\theta-\frac{1}{2}\sinh\theta\cosh\theta$$

$$=-\frac{1}{2}\ln(\cosh\theta+\sinh\theta)-\frac{1}{2}\sinh\theta\cosh\theta$$

$$=-\frac{1}{2}\ln(\csc x+\cot x)-\frac{1}{2}\cot x\csc x$$

$$=\frac{1}{2}\ln(\csc x-\cot x)-\frac{1}{2}\cot x\csc x.$$

Example 3. $\int\tan^2 x\sec x\,dx=\int\cos^{-3}x\sin^2 x\,dx$ $(m=-3,\ n=2,\ m+n=-1)$. This integral may be rewritten as

$$\int\frac{\tan^2 x\sec^2 x\,dx}{\sec x}=\int\frac{\tan^2 x\,d(\tan x)}{\sec x}.$$

Using substitution set I, we obtain

$$\int\frac{\sinh^2\theta\cosh\theta\,d\theta}{\cosh\theta}=\int\sinh^2\theta\,d\theta=\frac{1}{2}\int(\cosh 2\theta-1)d\theta$$

$$= \frac{1}{4}\sinh 2\theta - \frac{1}{2}\theta = \frac{1}{2}\sinh\theta\cosh\theta - \frac{1}{2}\ln(\cosh\theta + \sinh\theta)$$

$$= \frac{1}{2}\tan x \sec x - \frac{1}{2}\ln(\sec x + \tan x)$$

$$= \frac{1}{2}\tan x \sec x + \frac{1}{2}\ln(\sec x - \tan x).$$

Example 4. $\displaystyle\int \frac{dx}{\sin^2 x \cos x}$ $\qquad (m = -1,\ n = -2,\ m+n = -3).$

This integral may be rewritten as

$$\int \frac{\sec^3 x\, dx}{\tan^2 x} = \int \frac{(\sec x)(\sec^2 x\, dx)}{\tan^2 x} = \int \frac{\sec x\, d(\tan x)}{\tan^2 x}.$$

Using substitution set I, we obtain

$$\int \frac{(\cosh\theta)(\cosh\theta\, d\theta)}{\sinh^2\theta} = \int \coth^2\theta\, d\theta = \int (\operatorname{csch}^2\theta + 1)d\theta$$

$$= -\coth\theta + \theta = \frac{-\cosh\theta}{\sinh\theta} + \theta$$

$$= \frac{-\cosh\theta}{\sinh\theta} + \ln(\cosh\theta + \sinh\theta)$$

$$= \frac{-\sec x}{\tan x} + \ln(\sec x + \tan x)$$

$$= -\csc x + \ln(\sec x + \tan x).$$

(Constants of integration have been omitted in all cases in the interest of brevity.)

The present writer does not know who originated this very ingenious method. In its imperfect form, it was taught to several generations of engineering students at Stevens Institute of Technology by the late Professor Charles O. Gunther, who learned it from his predecessor, Professor J. Burkitt Webb. It was included in a text-book entitled "Integration by Trigonometric and Imaginary Substitution" by Professor Gunther, published by Van Nostrand in 1907. This book never had a large sale and has long been out of print.

Professor Gunther called the method "imaginary substitution," because he used the substitution

$$\tan x = i \sin\theta,$$

from which he obtained $d(\tan x) = i \cos\theta\, d\theta$, and

$$\sec x = \sqrt{(1 + \tan^2 x)} = \sqrt{(1 - \sin^2\theta)} = \cos\theta.$$

As applied to example 1 above, these substitutions work out as follows:

$$\int \sec x\,dx = \int \frac{\sec^2 x\,dx}{\sec x} = \int \frac{d(\tan x)}{\sec x} = \int \frac{i\cos\theta\,d\theta}{\cos\theta} = \int i\,d\theta = i\theta.$$

Since $e^{i\theta} = \cos\theta + i\sin\theta$,

$$i\theta = \ln(\cos\theta + i\sin\theta) = \ln(\sec x + \tan x).$$

This illustrates that the mechanics and the results of the process, as used by Professor Gunther, are the same as described here.

A GENERALIZATION OF THE VON KOCH CURVE

JOEL E. SCHNEIDER, University of Oregon

The von Koch curve is a plane curve which, while of infinite length, encloses a simply connected region of finite area. The von Koch curve is constructed in the following manner: given an equilateral triangle, trisect each side, construct equilateral triangles on each middle segment so that the interiors of the added triangles lie in the exterior of the base triangle, and delete the segments upon which triangles are constructed. Repeat the process of trisection of sides and addition of triangles, *ad infinitum.*

Let C_n denote the curve when n sets of triangles have been added and let C_n' be its perimeter. It is easy to see that $C_n' = (4/3)C_{n-1}' = (4/3)^n C_0'$ and that therefore C_n' is unbounded as $n \to \infty$. The area C_n'' of the region enclosed by C_n is the sum of the areas of the added triangles and of the original triangle. At the nth stage, $n > 0$, $3 \cdot 4^{n-1}$ triangles are added. It is easy to see that if T_n is one of the triangles added at the nth stage, then $T_n'' = (1/9)^n T_0''$, where T_0 is the original triangle. Therefore,

$$C_n'' = T_0'' + \sum_1^n (3)(4^{i-1})(1/9)^i T_0''.$$

As $n \to \infty$, $C_n'' \to (8/5)T_0''$. Hence the region bounded by the von Koch curve has finite area, but the curve has infinite length.

There are at least three ways in which a generalization of the von Koch curve might be approached: by using a regular polygon other than an equilateral triangle, by employing a sectioning other than that of trisection, or by establishing an analogue in dimensions other than two.

It can be shown that the use of a regular polygon other than an equilateral triangle or a square as the basic figure gives rise to a curve whose interior is self-intersecting and therefore not simply connected. The construction of the curve based on the square is the same as that for the von Koch curve with "square" read in place of "triangle." The formulas for the perimeter of the resultant poly-

As applied to example 1 above, these substitutions work out as follows:

$$\int \sec x\,dx = \int \frac{\sec^2 x\,dx}{\sec x} = \int \frac{d(\tan x)}{\sec x} = \int \frac{i\cos\theta\,d\theta}{\cos\theta} = \int i\,d\theta = i\theta.$$

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With respect to the second approach to generalization, the construction of the curve may be stated as follows: given an equilateral triangle or a square, $(2k+1)$-sect each side, where k is a positive integer. On each side, number the segments consecutively from one endpoint of the side to the other. Construct the basic polygon on each even numbered segment so that the interiors of the added polygons lie in the exterior of the base polygon and delete the segments upon which polygons were constructed (see figure one). Repeat the process of $(2k+1)$-section and addition of polygons, *ad infinitum.*

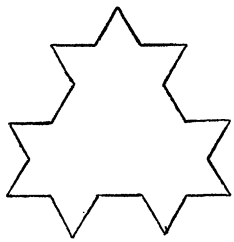

FIG. 1. $C_1(2)$.

Considering the case in which the basic polygon is an equilateral triangle, let $C_n(k)$ denote the polygon of the construction when n sets of triangles have been added and let $C_n'(k)$ be its perimeter. After sectioning, k triangles are constructed on each side of $C_{n-1}(k)$ and that number of segments is deleted. Thus $C_n(k)$ has $3k+1$ times as many sides as $C_{n-1}(k)$, each of which has a length $1/(2k+1)$ times that of a side of $C_{n-1}(k)$. Thus $C_n'(k)=[(3k+1)/(2k+1)]^n C_0'(k)$. Since $k>0$, $3k+1>2k+1$. Then, as $n\to\infty$, $C_n'(k)\to\infty$.

Let $C_n''(k)$ denote the area of the region enclosed by $C_n(k)$. Then $C_n''(k)$ is the sum of the areas of the added triangles and the original triangle. At the nth stage, $n>0$, k triangles are added for each side of $C_{n-1}(k)$. Each of the triangles added at this stage has an area $1/(2k+1)^2$ times the area T_0'' of the original triangle T_0. Then,

$$\begin{aligned} C_n''(k) &= T_0'' + \sum_1^n k[3(3k+1)^{i-1}][1/(2k+1)^2]^i T_0'' \\ &= T_0''\left\{1 + 3k/(3k+1)\sum_1^n [(3k+1)/(2k+1)^2]^i\right\}. \end{aligned}$$

Since $k>0$, $3k+1<(2k+1)^2$. Thus, as $n\to\infty$,

$$C_n''(k)\to[4(k+1)/(4k+1)]T_0''.$$

If the basic polygon is a square, then the perimeter is again unbounded while the area approaches $(k+1)/k$ times that of the original square.

In establishing an r-dimensional analogue to the von Koch curve, a regular r-simplex is used as the basic figure. In constructing the analogue, it is necessary to divide the bounding $(r-1)$-simplexes into mutually congruent regular $(r-1)$-simplexes. It can be shown that a division of a regular k-simplex into mutually congruent regular k-simplexes gives rise to a division of its bounding t-simplexes, $1 \leqq t \leqq k$, into mutually congruent regular t-simplexes. It can also be shown that no such division of a regular 3-simplex is possible. Therefore, there is no such division of a regular k-simplex for $k > 3$. Hence the analogue is restricted to two and three dimensions.

The three dimensional analogue is constructed as follows: given a regular tetrahedron, k^2-sect each of the faces, where k is a positive integer, $k > 1$. (It can be shown that if an equilateral triangle is to be divided into a number of mutually congruent equilateral triangles, then that number is a perfect square.) On each face choose each of the segments which, under a translation only, is *not* directly congruent to one of the segments which includes a vertex of the face (see figure two). On each of the chosen segments construct a regular tetrahedron so that the interiors of the added tetrahedra lie in the exterior of the base tetrahedron and delete the segments upon which the tetrahedra are constructed. Repeat the processes of sectioning and construction, *ad infinitum.*

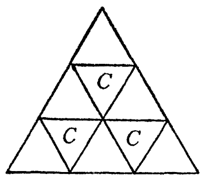

FIG. 2. One sectioned face of $P_n(3)$. Tetrahedra are constructed on the segments marked "C".

Let $P_n(k)$ be the polyhedron of the construction when n sets of tetrahedra have been added and let $P_n'(k)$ be the area of its surface. After the sectionings, there are k^2 segments on each face of $P_{n-1}(k)$. On $k(k-1)/2$ of these, tetrahedra are constructed, and this number of segments is deleted. Thus $P_n(k)$ has $k(2k-1)$ times as many faces as $P_{n-1}(k)$, each of which has an area $1/k^2$ times that of a face of $P_{n-1}(k)$. Hence, $P_n'(k) = [(2k-1)/k]^n P_0'(k)$. Since $k > 1$, $2k-1 > k$. Then, as $n \to \infty$, $P_n'(k) \to \infty$.

Let $P_n''(k)$ be the volume of the region enclosed by $P_n(k)$. Then $P_n''(k)$ is the sum of the volumes of the added tetrahedra and of the original tetrahedron. At the nth stage, $n > 0$, $k(k-1)/2$ times as many tetrahedra are added as there are faces of $P_{n-1}(k)$ and each of these added tetrahedra has a volume $1/k^3$ times the volume T_0'' of the original tetrahedron T_0. Then,

$$P_n''(k) = T_0'' + \sum_1^n [k(k-1)/2]\{4[k(2k-1)]^{i-1}\}[1/k^3]^i T_0''$$

$$= T_0'' \left\{1 + [2(k-1)/(2k-1)] \sum_1^n [(2k-1)/k^2]^i \right\}.$$

For $k > 1$, $2k-1 < k^2$. Then as $n \to \infty$, $P_n''(k) \to [(k+1)/(k-1)]T_0''$.

References

1. E. Kasner and J. Newman, Mathematics and the Imagination, Simon and Schuster, New York, 1940.

2. H. von Koch, Une méthode géométrique élémentaire pour l'étude de certaines questions de la théorie des courbes planes, Acta Math., 30 (1906) 145–176.

INVERSION WITH RESPECT TO THE CENTRAL CONICS

NOEL A. CHILDRESS, University of Mississippi

The purpose of this paper is to generalize to some extent inversion with respect to the circle in the real Euclidean plane.

DEFINITION 1. *The point P' is the image of a point P in an inversion with respect to a central conic if P' lies on the line OP and $OP\cdot OP'=a^2b^2(1+m^2)/(b^2+ka^2m^2)$, where O is the center of the central conic, a and b are respectively the semi-major and semi-minor axes, m is the slope of the line OP measured from the principal axis of the conic, and $k^2=1$. If m does not exist, $OP\cdot OP'=kb^2$. The image of O is the ideal point, and conversely. The point O is called the center of inversion. The inversion is with respect to an ellipse or a hyperbola according as $k=1$ or $k=-1$, respectively. This conic will be referred to as the central conic of inversion; i.e., ellipse of inversion or hyperbola of inversion.*

For simplicity suppose the central conic has the equation $x^2/a^2+ky^2/b^2=1$. If $P(x, y)$ has for its image $P'(x', y')$, $OP\cdot OP'=xx'(1+m^2)$ by use of the distance formula and $y/x=m=y'/x'$ assuming m exists. By Definition 1, $xx'(1+m^2)=a^2b^2(1+m^2)/(b^2+ka^2m^2)$ and it follows that $x'=a^2b^2x/(b^2x^2+ka^2y^2)$ using again $m=y/x$. Solving for y' using $y'=mx'$ and $y=mx$ one obtains

$$\text{(T)} \qquad x' = a^2b^2x/(b^2x^2 + ka^2y^2), \qquad y' = a^2b^2y/(b^2x^2 + ka^2y^2).$$

In case m does not exist, this same procedure gives $x'=0$ and $y'=kb^2/y$. But m not existing means $x=0$ and T reduces to the above equations under these circumstances. It is readily seen that T is an involution of period two. Since $b^2x^2+ka^2y^2=a^2b^2$ for points on the conic mentioned previously, points of the conic of inversion are invariant under T.

DEFINITION 2. *The curve C' is the image of a curve C in an inversion with respect to a central conic if C' consists of the totality of points which are the images of the points of C. If C passes through the center of inversion O, the image of O on C' will be the ideal point on the tangent line to C at O.*

THEOREM 1. *The inverse of a line through the center of inversion is the same line except in the case when $k=-1$ and the line is an asymptote of the hyperbola of inversion.*

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THEOREM 1. *The inverse of a line through the center of inversion is the same line except in the case when $k=-1$ and the line is an asymptote of the hyperbola of inversion.*

Proof. Applying T to $y=mx$ gives $y=mx$ provided $m=b/a$ and $k=-1$ do not hold simultaneously and applying to $x=0$ gives $x=0$.

DEFINITION 3. *If two ellipses or two hyperbolas have parallel axes and have equal eccentricities, then they are said to be of the same semi-form. If in addition the principal axes are parallel, then they are said to be of the same form.*

THEOREM 2. *The inverse of a line not through the center of inversion is a central conic passing through the center of inversion of the same form as the central conic of inversion, and conversely.*

Proof. Applying T to the line $y=Ax+B(B\neq 0)$ gives $x^2/a^2+ky^2/b^2+Ax/B-y/B=0$, and to $x=C(C\neq 0)$ gives $x^2/a^2+ky^2/b^2-x/C=0$. The result follows in each case. Conversely, applying T to the central conic $x^2/a^2+ky^2/b^2+Dx+Ey=0$ (D and E not both zero) gives the line $Dx+Ey+1=0$, which is the indicated result. It is easily established that the ideal point on this line is the image of O.

THEOREM 3. *The inverse of a central conic of the same semi-form as the central conic of inversion and not passing through the center of inversion is another central conic of the same form.*

Proof. Let the conic be $x^2/a^2+ky^2/b^2+Dx+Ey+F=0(F\neq 0)$. Applying T to this equation and simplifying gives $x^2/a^2+ky^2/b^2+Dx/F+Ey/F+1/F=0$. This implies the indicated result.

THEOREM 4. *The inverse of any conic not of the same semi-form as the central conic of inversion and passing through the center of inversion is a cubic curve with a singular point at the center of inversion.*

Proof. Applying T to the conic $Ax^2+Bxy+Cy^2+Dx+Ey=0(A=1/a^2, B=0,$ and $C=k/b^2$ cannot hold simultaneously) and simplifying gives $Aa^2b^2x^2+Ba^2b^2xy+Ca^2b^2y^2+Db^2x^3+Eb^2x^2y+kDa^2xy^2+kEa^2y^3=0$. It is easily established by the usual procedure that this cubic has a singular point at the origin.

THEOREM 5. *The inverse of any conic not of the same semi-form as the central conic of inversion and not passing through the center of inversion is a curve of the fourth degree with a singular point at the center of inversion.*

Proof. Applying T to the conic $Ax^2+Bxy+Cy^2+Dx+Ey+F=0(A=1/a^2,$ $B=0$, and $C=k/b^2$ cannot hold simultaneously and $F\neq 0$) and simplifying gives $Aa^4b^4x^2+Ba^4b^4xy+Ca^4b^4y^2+Da^2b^4x^3+Ea^2b^4x^2y+kDa^4b^2xy^2+kEa^4b^2y^3+Fb^4x^4+2kFa^2b^2x^2y^2+Fa^4y^4=0$. Also, it can be easily established that this curve has a singular point at the origin.

COROLLARY. *If the conic in Theorem 4 or in Theorem 5 is an ellipse, a parabola, or a hyperbola, the singular point of the image curve is an isolated point, a cusp, or a double point, respectively.*

Proof. For either image curve, the tangents at the singular point have the equation $Ax^2+Bxy+Cy^2=0$. Hence, the singular point is an isolated point, a cusp, or a double point according as B^2-4AC is negative, zero, or positive, respectively.

Many other facts concerning inversion with respect to the central conics may be established by this method. For instance, if the central conic of inversion is an ellipse, a point inside this conic has its image outside, and conversely. In the case of the hyperbola of inversion, a point interior to an angle formed by the asymptotes with a branch of the hyperbola has its image inside the hyperbola if the original point is outside, and conversely, a point on an asymptote has for its image the ideal point on that asymptote, and otherwise the center of inversion is between a point and its image. Familiar properties of inversion with respect to a circle result when $a=b$ and $k=1$, and inversion with respect to the equilateral hyperbola can be studied by letting $a=b$ and $k=-1$. Also, an extension into the complex plane is most interesting.

NOTE ON A COMBINATORIAL IDENTITY

H. J. HIETALA, Silverado, California AND B. B. WINTER, Costa Mesa, California

0. The purpose of this note is to exhibit two proofs of the identity

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j+1}}{j}=\sum_{j=1}^{n}\frac{1}{j}\,. \tag{1}$$

A direct algebraic proof is given in section 2 of the note. In section 1, we give a probabilistic proof based on some reliability considerations which first brought this identity to our attention. The result can also be obtained from some other combinatorial identities derived in [2] by probabilistic considerations.

1. Consider a system of n devices "in parallel"—i.e., used so that the system is considered failed when all n devices have failed. Assume that the devices fail independently and that the time-to-failure, for any one of them, is distributed exponentially with mean θ. Then the distribution G of time to system failure is

$$G(x) = [1 - e^{-x/\theta}]^n$$

and the expected time to system failure is (see, e.g., [**3**], p. 211)

$$\begin{aligned}\int_0^\infty [1 - G(x)]\,dx &= \int_0^\infty\left[1-\sum_{j=0}^{n}\binom{n}{j}(-e^{-x/\theta})^j\right]dx \\ &= \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\int_0^\infty e^{-x/(\theta/j)}\,dx \\ &= \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\frac{\theta}{j}\end{aligned} \tag{2}$$

since $\int_0^\infty e^{-x}dx=1$.

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since $\int_0^\infty e^{-x}dx=1$.

On the other hand, the time to system failure can be thought of as the nth order statistic in a sample of size n from an exponential distribution with mean θ. It is shown below that the expectation of the rth order statistic in such a sample is

$$\theta \sum_{j=1}^{r} \frac{1}{n-j+1}. \tag{3}$$

With $r=n$, (3) becomes

$$\theta \sum_{j=1}^{n} \frac{1}{n-j+1} = \theta \sum_{j=1}^{n} \frac{1}{j}. \tag{4}$$

Equating (4) with the last term of (2) and cancelling θ proves (1).

The result (3) was proved by Gumbel [1] and, independently, by Epstein [2]. The following proof of (3) was suggested to us by Epstein as a simplification of the proof he published in [2]. Let $t_{r,n}$ be the rth order statistic in a sample of size n from an exponential distribution with mean θ. Then

$$t_{r,n} = t_{1,n} + (t_{2,n} - t_{1,n}) + \cdots + (t_{r,n} - t_{r-1,n}). \tag{5}$$

But $t_{j,n}$ may be thought of as the time to the jth failure in life-testing n components, each with an exponential failure distribution with mean θ. Then the occurrence of the first failure is governed by a stochastic process which is the superposition of n Poisson processes with mean θ. Therefore the process is a Poisson process with mean θ/n and

$$E(t_{1,n}) = \theta/n.$$

After the first failure has occurred, $n-1$ components remain on test. Therefore the occurrence of the second failure is governed by a Poisson process with mean $\theta/(n-1)$ and, since $t_{2,n}-t_{1,n}$ is the time from the first to the second failure,

$$E(t_{2,n} - t_{1,n}) = \theta/(n-1).$$

Similarly,

$$E(t_{j,n} - t_{j-1,n}) = \theta/(n-j+1).$$

Thus, in view of (5),

$$E(t_{r,n}) = \theta \sum_{j=1}^{r} \frac{1}{n-j+1}.$$

2. Define S_n and D_n as follows:

$$S_0 = 0,\ S_n = \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^{j+1}}{j} \qquad \text{for } n = 1, 2, \cdots.$$

$$D_n = S_n - S_{n-1} - \binom{n}{n} \frac{(-1)^{n+1}}{n} \qquad \text{for } n = 1, 2, \cdots.$$

Clearly, (1) holds if and only if

$$S_n - S_{n-1} = \frac{1}{n}$$

i.e., if and only if

$$D_n = \frac{1}{n}\,[1 + (-1)^n]. \tag{6}$$

To prove (6), substitute the expressions for S_n and S_{n-1} into the defining equation of D_n, obtaining

$$D_n = \sum_{j=1}^{n-1} \binom{n}{j} \frac{(-1)^{j+1}}{j} - \sum_{j=1}^{n-1} \binom{n-1}{j} \frac{(-1)^{j+1}}{j}.$$

But

$$\frac{1}{j}\left[\binom{n}{j} - \binom{n-1}{j}\right] = \frac{1}{j}\binom{n-1}{j}\frac{j}{n-j} = \frac{1}{n}\binom{n}{j};$$

therefore,

$$\begin{aligned} D_n &= \frac{1}{n}\sum_{j=1}^{n-1}\binom{n}{j}(-1)^{j+1} \\ &= \frac{1}{n}\left[1 + (-1)^n + \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}\right] \\ &= \frac{1}{n}\,[1 + (-1)^n] \end{aligned}$$

since

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1} = -(1-1)^n = 0.$$

This proves (6) and thus (1).

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CONIC POWERS OF POINT SETS

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In Figure 1, P represents any fixed point at a distance R from the center C of a fixed circle of radius r. Noticing that two triangles are equiangular and therefore similar, we see that $PA:PL=PM:PB$ or $PA\cdot PB=PL\cdot PM$. As the right-hand side of the last equation does not involve points A or B, $PA\cdot PB$ is a constant, $(R+r)(R-r)=R^2-r^2$.

THEOREM. *The product of any two collinear distances of a given point from a given circle is constant.*

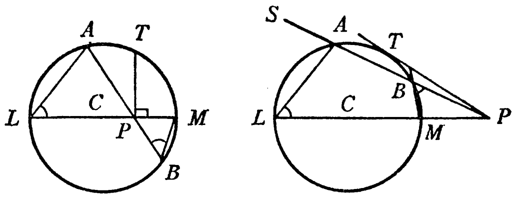

FIG. 1.

Although this theorem summarizes the content of two theorems known to the classical Greeks, the constant R^2-r^2 was first considered as a potent geometric tool by Gaultier [1] 150 years ago. According to Steiner's usage, it is called the "power of the point with respect to the circle." It follows at once that

$$R^2 - r^2 = (x_0 - h)^2 + (y_0 - k)^2 - r^2 = f(x_0, y_0) \tag{1}$$

is the power of point (x_0, y_0) with respect to the circle $f(x, y)=0$.

A wealth of theorems concerning this property are studied in college geometry. Perhaps more important is its use there as an instrument of discovery. As Court [2] devotes 27 pages to the topic, we shall limit our comments to these:

(a) The power of a point with respect to a circle is positive, zero, or negative according as the point lies outside, on, or inside the circle. (In all its extensions it enables a curve and a reference point C to divide the plane into signed parts.)

(b) In either figure its absolute value is $(PT)^2$.

(c) The power of the center of a circle with respect to the circle is $-r^2$.

(d) The philosophically inclined may ponder on the possible value when the line PS, however far produced, does not meet the circle.

Although the Vandermonde determinant,

$$\left| x_0^2 + y_0^2\ x_1\ y_2\ 1_3 \right| = \begin{vmatrix} x_0^2 + y_0^2 & x_0 & y_0 & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix}, \tag{2}$$

vanishes when and only when the set of four points involved lie on a common circle, it is not a good measure of the departure from "circularity" of the set.

This is because it is not in a normalized form; it contains an irrelevant factor, a fault common to Vandermonde determinants.

However, it is proved in the theory of equations [3] that

$$| x_0^2 + y_0^2 \; x_1 \; y_2 \; 1_3 | = 2K(R^2 - r^2), \tag{3}$$

where (x_0, y_0) is the given point; the given circle is circumscribed through (x_1, y_1), (x_2, y_2), (x_3, y_3); and K is the signed area of the inscribed triangle.

THEOREM. *Given a set of four points in a plane, the product of the power of any such point with respect to the circle circumscribed through the other three and the signed cyclic area of the inscribed triangle is constant.*

This is a symmetry that the Greeks, for lack of the determinantal machinery, could not achieve.

A common complaint of the mathematics major is that he can find no topic in which to do original work. As a former teacher, the present writer had always recommended that the student generalize some common topic, as this one. Here one might ask: Why not the power of a point with respect to a hypocycloid? Indeed, let us promise ourselves that we will produce the power with respect to any curve before this paper is done.

As a first step, let us make the obvious improvement by defining the "relative" or dimensionless power as

$$(R^2 - r^2)/r^2 = \tfrac{1}{2} | x_0^2 + y_0^2 \; x_1 \; y_2 \; 1_3 | /(r^2 K). \tag{4}$$

Experiments with the other simplest conic, a rectangular hyperbola (with horizontal and vertical asymptotes), will reveal that $R^2 - r^2$ is not now a constant as PS rotates about P; the theorem quoted at the beginning of this article holds only for circles. At this point in any investigation, one must be flexible, not rigidly holding to some preconceived notion. In generalizing the concept to other conics it has not been realized that PS must be taken as passing through the center C of the conic, as the notations R and r now truly imply (CP and $CL = CM$ respectively in Figure 2).

Analogues of the second theorem are valid; to illustrate we show the formula used in computing the "power of a point with respect to a rectangular hyperbola,"

$$| 2x_0y_0 \; x_1 \; y_2 \; 1_3 | = 2K(R^2 - r^2) \sin 2\theta \tag{3'}$$

or

$$(R^2 - r^2)/r^2 = \tfrac{1}{2} | 2x_0y_0 \; x_1 \; y_2 \; 1_3 | /(a^2 K). \tag{4'}$$

Again K is the signed area of the inscribed triangle and a is the length of the transverse semi-axis. θ is the angle CP makes with an asymptote.

However, it is not more difficult to derive the "power of a point with respect to a general conic" after having developed the idea of a normalized power with respect to a center. We shall take $f(x, y)$ to represent, indifferently, either the Vandermonde determinant, the similarly normalized conic function, or its bi-linear matrix form as advocated by Amir-Moéz [4], all equivalently determined by the five points $P_1(x_1, y_1), \cdots, P_5(x_5, y_5)$ of Figure 2:

$$
(2') \qquad
\begin{aligned}
f(x, y) &= \left| x^2 \; x_1y_1 \; y_2^2 \; x_3 \; y_4 \; 1_5 \right| \\
&= Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F \\
&= [x \;\; y \;\; 1]\begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.
\end{aligned}
$$

The center, $C(h, k)$, of the conic $f(x, y)=0$ is [5],

$$
h = \begin{vmatrix} D & B \\ E & C \end{vmatrix} \Big/ \delta, \qquad k = \begin{vmatrix} A & D \\ B & E \end{vmatrix} \Big/ \delta,
$$

where

$$
\delta = -\begin{vmatrix} A & B \\ B & C \end{vmatrix},
$$

is the discriminant, invariant under translations and rotations, that classifies the conic as of elliptic, parabolic, or hyperbolic type, according as it is positive, zero, or negative respectively.

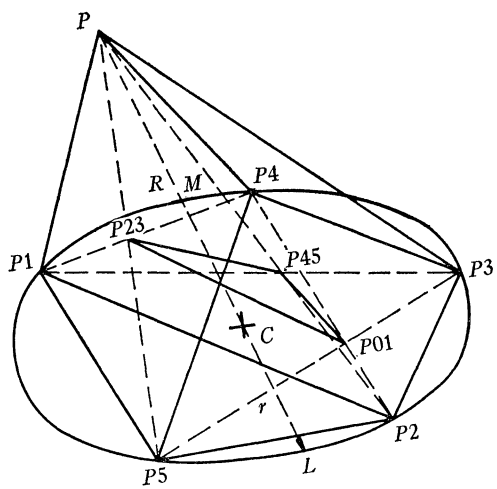

FIG. 2.

The center of a parabola ($\delta=0$) is at infinity; the absolute power is infinite but the relative power vanishes for any finite position of the fixed point. For this limiting case we shall elect merely to verify that the resultant formula will produce the proper relative value 0.

Assuming that we are dealing with a central conic, we see that the translation $x'=x-h$, $y'=y-k$ produces

$$f'(x', y') = Ax'^2 + 2Bx'y' + Cy'^2 + f(h, k) = 0, \tag{5}$$

where [5]

$$f(h, k) = -\Delta/\delta, \tag{6}$$

and

$$\Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

is the coefficient determinant, invariant under translations and rotations, and indicating (by its vanishing) when the conic is degenerate.

The equation of the line through the center C of the conic and the point $P(x_0, y_0)$ is now

$$y' = (y_0'/x_0')x'. \tag{7}$$

The conic (5) and line (7) intersect at L and $M((x', y'))$, where

$$x'^2 = x_0'^2\Delta/(\delta\phi), \qquad y'^2 = y_0'^2\Delta/(\delta\phi),$$
$$\phi = Ax_0'^2 + 2Bx_0'y_0' + Cy_0'^2 = f'(x_0', y_0') - f(h, k),$$

so that

$$(CL)^2 = (CM)^2 = r^2 = x'^2 + y'^2 = \Delta(x_0'^2 + y_0'^2)/(\delta\phi) = \Delta R^2/(\delta\phi) = \Delta(CP)^2/(\delta\phi).$$

Then $PL \cdot PM(R+r)(R-r) = R^2 - r^2 = \delta(\phi - \Delta/\delta)r^2/\Delta$ or

$$(R^2 - r^2)/r^2 = \delta f(x_0, y_0)/\Delta, \tag{8}$$

a formula also valid for the parabolic type.

By (6) and (8), the power of a point with respect to a conic finally reduces to the determinantal form

$$(R^2 - r^2)/r^2 = -f(x_0, y_0)/f(h, k). \tag{1'}$$

This is the power of point (x_0, y_0) with respect to the conic $f(x, y) = 0$ with center (h, k). When f is interpreted as the conic function, rather than as the determinant, it is not necessary, as it is in (8), that it be normalized.

When $n \geqq 6$, the "conic power of a set of n points" may be obtained from

$$[(R^2 - r^2)/r^2]^2 = [f'(x_0, y_0)f(x_0, y_0)]/[f'(h, k)f(h, k)],$$

where $f(x, y)$ is now restricted to its determinantal interpretation, and $f'(x, y)$ indicates the transpose. This should prove of use in numerical analysis, supplying a geometric interpretation, for example, of Moore's pseudo-inverse determinant of curve fitting practice.

Entirely geometric interpretations of the relative power of a point with respect to a conic may also be developed. To do so, we must recall two famous theorems of projective geometry.

THEOREM OF DESARGUES (1648). *If the 3 junctions of pairs of corresponding vertices of two triangles are concurrent, then the 3 intersections of the pairs of corresponding sides are collinear and the 6 intersections of the pairs of noncorresponding sides are conconic* [6].

Six points in a plane determine 60 distinct "Pascal" hexagons, as $PP_2P_4P_1P_3P_5$ of Figure 2. The three intersections of the pairs of opposite sides P_2P_4, P_3P_5; P_4P_1, P_5P; P_1P_3, PP_2 are the vertices of the "Veronese" [7] triangle $P_{01}P_{23}P_{45}$.

THEOREM OF PASCAL (1639). *If a hexagon is inscribed in a conic, the 3 intersections of the pairs of opposite sides lie on a line.*

In 1875 Hunyady [8, 9] established the equivalence of the two theorems. Together with a result supplied by Mertens [10] and a geometric interpretation supplied by Maley [11], there also ensues a generalization of each theorem.

EXTENDED THEOREM OF DESARGUES.

$$\text{(D)} \qquad f(x, y) = (2!)^4 \begin{vmatrix} (P\,P_5P_1)(P_2P_3P_5) & (P\,P_3P_4)(P\,P_3P_5) \\ (P_2P_4P_3)(P\,P_1P_4) & (P_1P_2P_5)(P_1P_2P_4) \end{vmatrix}$$

Here $f(x, y)$ represents the Vandermonde determinant of a conic and $(P_2P_3P_5)$, for example, is the signed area of a triangle.

To appreciate the second theorem it is necessary to notice, as Hunyady and Mertens apparently did not, that the Pascal hexagon $PP_2P_4P_1P_3P_5$ consists of 3 "Hunyady" quadrilaterals $PP_1P_2P_3$, $P_2P_3P_4P_5$, $P_4P_5P_1P$, characterized by sharing one side, each with each. The pairs of opposite sides of the hexagon are the pairs of diagonals of the quadrilaterals; the diagonal intersections are the vertices of the Veronese triangle. (At the time [11] was written, the present writer was unaware of Hunyady's work; to make amends he has taken the liberty of naming these quadrilaterals in this pioneer's honor.)

EXTENDED THEOREM OF PASCAL.

$$\text{(P)} \qquad f(x, y) = (2!)^4(P_{01}P_{23}P_{45})(PP_1P_2P_3)(P_2P_3P_4P_5)(P_4P_5P_1P).$$

(P) and (1′) then immediately lead to the geometric result,

$$(P_{01}P_{23}P_{45})(PP_1P_2P_3)(PP_4P_5P_1) = [(r^2 - R^2)/r^2](P_{01}C_{23}C_{45})(CP_1P_2P_3)(CP_4P_5P_1),$$

the relative power in terms of the areas of the two Veronese triangles and four of the Hunyady quadrilaterals associated with the two Pascal configurations of points P_1, P_2, P_3, P_4, P_5 and either point P or C, the fixed point and the center respectively.

Our geometric definition, $(R^2-r^2)/r^2$, of the relative power of a point with respect to a conic presents an obstacle to further progress of moment, in that it implies a central curve, as the hypocycloid. However, the last relation above shows that there are many possible geometric interpretations for the conic alone ((D) and (1′) would lead to another). Hence, one is led to an analytic definition that imposes no such restraint to the type of curve.

DEFINITION. *The relative power of a pair of points,* (x, y) *and* (X, Y), *with respect to a curve* $f(x, y)=0$ *is the canonical form,*

$$-f(x, y)/f(X, Y). \tag{1''}$$

For straight lines, this becomes simply the ratio of the distances of the two points to the line, positive if the line passes between them. The negative sign has the same significance as that associated with voltages—both Gaultier and Franklin made unfortunate sign conventions.

In view of a previous statement that the student should always be able to extend and generalize, it is imperative to prove that this paper at least is incomplete. The radix conic, $f'(h', k')f(x, y)=f(h, k)f'(x, y)$, probably has harmonic properties, not investigated here. Hint: first show that the radix circle of two circles passes through their four intersections and through the two intersections of their two pairs of common exterior and interior tangents. The last two of these points are shared with the central "line," which also contains the four poles of the imaginary and other common chord. Then dualize.

The usual and obvious generalization to higher dimension is too easy and has already been started. Spherical powers are already in existence. Certainly, z's may be added to formula (1″).

Without interesting geometric interpretations of the Vandermonde determinants of higher degrees and dimension (now lacking) in the manner of Hunyady, the general definition (1″) lacks savor.

Consider the general cubic Vandermonde determinant

$$\left| x_0^3 \; x_1^2y_1 \; x_2y_2^2 \; y_3^3 \; x_4^2 \; x_5y_5 \; y_6^2 \; x_7 \; y_8 \; 1_9 \right|.$$

Of degree 20 (product of 10 areas?), it is determined by 10 points. Now, a decagon $P_0P_1P_2P_3P_4P_5P_6P_7P_8P_9$ is resolvable to 6 Hunyady pentagons, $P_0P_1P_2P_3P_4$, $P_0P_1P_5P_7P_6$, $P_1P_2P_8P_9P_5$, $P_2P_3P_7P_6P_8$, $P_3P_4P_9P_5P_7$, $P_0P_4P_9P_8P_6$, characterized as sharing one side, each with each. Their six "diagonal centers" determine a conic Vandermonde, i.e., three Hunyady quadrilaterals whose diagonal centers determine a linear Vandermonde, i.e., a Veronese triangle, for a total of ten areas.

Is it true, for example, that the set of six points is conconic, and the set of three points collinear, if and only if the set of ten points is concubic? To establish this, a proper definition of the "diagonal center" of a pentagon (or, preferably, of the general polygon!) is in order. . . .

A N.Y.S. private citizen research project, aided by computer time donated by the Computing Center, SUNY at Buffalo.

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NOTES

TIEN-HSUNG LIN, Formosa, China

1. A proof of $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$.

(i) Let $AD=1$ as shown in Fig. 1(A).

Then $\sin(\alpha+\beta)=AC$. Furthermore, $DE=\sin\beta$, $AE=\cos\beta$, and $BE=\cot\alpha\cdot DE=\cot\alpha\sin\beta$;

$$\sin\alpha = \frac{AC}{AB} = \frac{AC}{AE+BE} = \frac{\sin(\alpha+\beta)}{\cos\beta+\cot\alpha\sin\beta},$$

$$\sin(\alpha+\beta) = \sin\alpha(\cos\beta+\cot\alpha\sin\beta) = \sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

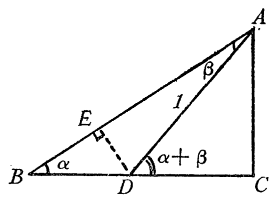

FIG. 1(A).

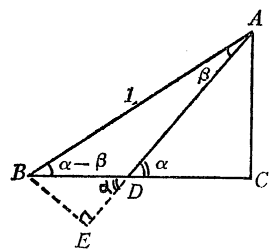

FIG. 1(B).

(ii) Let $AB=1$ as shown in Fig. 1(B).

Then $\sin(\alpha-\beta)=AC$. Furthermore, $BE=\sin\beta$, $AE=\cos\beta$, and $DE=\cot\alpha\cdot BE=\cot\alpha\sin\beta$;

$$\sin\alpha = \frac{AC}{AD} = \frac{AC}{AE-DE} = \frac{\sin(\alpha-\beta)}{\cos\beta-\cot\alpha\sin\beta},$$

$$\therefore\ \sin(\alpha-\beta) = \sin\alpha(\cos\beta-\cot\alpha\sin\beta) = \sin\alpha\cos\beta-\cos\alpha\sin\beta.$$

Similarly, $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$ may be derived in the same way.

2. If I is the incenter of triangle ABC as shown in Fig. 2, then $AI>ID$, $BI>IE$, and $CI>IF$.

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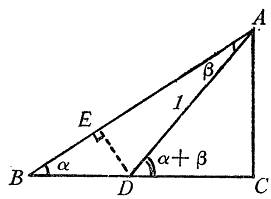


FIG. 1(A).

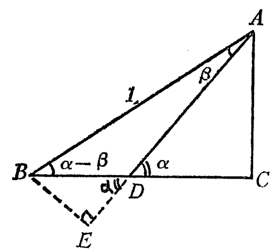


FIG. 1(B).

(ii) Let $AB = 1$ as shown in Fig. 1(B).

Then $\sin(\alpha-\beta) = AC$. Furthermore, $BE = \sin \beta$, $AE = \cos \beta$, and $DE = \cot \alpha \cdot BE = \cot \alpha \sin \beta$;

$$\sin \alpha = \frac{AC}{AD} = \frac{AC}{AE - DE} = \frac{\sin(\alpha - \beta)}{\cos \beta - \cot \alpha \sin \beta},$$

$$\therefore \ \sin(\alpha - \beta) = \sin \alpha(\cos \beta - \cot \alpha \sin \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

Similarly, $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$ may be derived in the same way.

2. If I is the incenter of triangle ABC as shown in Fig. 2, then $AI > ID$, $BI > IE$, and $CI > IF$.

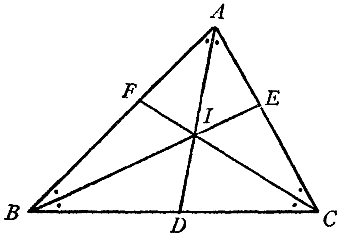


FIG. 2.

$$\text{In } \triangle ABD,\ \frac{AI}{ID} = \frac{AB}{BD}, \quad \text{and in} \quad \triangle ACD,\ \frac{AI}{ID} = \frac{AC}{CD}.$$

Hence

$$\frac{AI}{ID} = \frac{AB}{BD} = \frac{AC}{CD} = \frac{AB+AC}{BD+CD} = \frac{AB+AC}{BC} > 1.$$

Thus $AI > ID$. Similarly, we can show that $BI > IE$, $CI > IF$.

ANSWERS

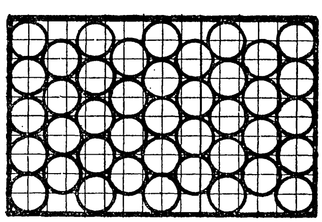
A359. By the Schwartz inequality

$$\int_0^1 F'(x)^2\,dx \int_0^1 dx \geqq \left\{ \int_0^1 F'(x)\,dx \right\}^2 = 1.$$

A360. All three of the approximations involve the proposition that $\sqrt{288}$ is approximately equal to $\sqrt{289}$.

A361. $F(\theta)$ denotes the distance the C. G. of a sector of angle 2θ is from the center. Consequently $F(\theta) = \sin\theta/2\theta$.

A362. See the figure below.



A363. If w is a primitive cube root of unity, it follows immediately that $w^{3m+2} + w^{3n+1} = 0$. Consequently x^2+x+1 is a factor of $x^{3m+2}+x^{3n+1}+1$. To find other factors, just divide.

(Quickies on page 188)

ANGLE PARTITION

WARREN A. REES, Los Angeles State College

It has long been known that Euclid's ruler and compass lack something in versatility in the partition of angles into an integral number of equal parts.

Much has been written about angle trisections and "angle trisectors," but little seems available on methods of n-section for all n.

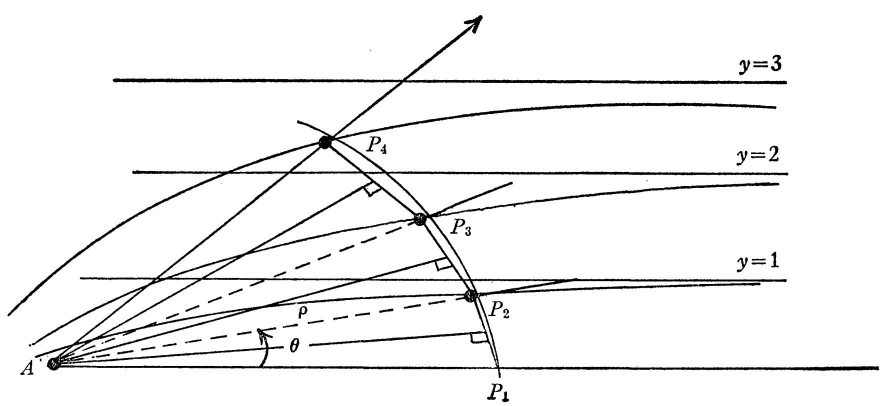
In a conversation with Mr. A. A. Treitman, a student of mine at Los Angeles State College, concerning the trisection of an angle by means of a carpenter's square, the author devised the following linkage which enables one to n-sect any angle by drawing a single arc of a circle and noting its intersection with certain curves whose polar equations are given by:

$$\rho = \frac{\sum_{i=1}^{n} \sin\left(\frac{2i-1}{2}\right)\theta}{1 - \cos n\theta}. \tag{n}$$

These curves are neatly constructed by means of a linkage called a *rake.* A rake consists of a set of n T-squares, linked at the ends of unit segments called *heads*. Each head has its perpendicular bisector as a handle.

If the rake is so placed that the handles all intersect at the vertex of A, which is the angle to be n-sected, and the initial end P_1 of the head of the first T-square moves on the initial side of angle A, while the handles move through A, each of the points $P_2, P_3, \cdots, P_n$ traces a curve whose equation is given by (n) and these curves are asymptotic to the lines, $y=1, 2, 3, \cdots, (n-1)$.

It is now clear that if the terminal side of angle A cuts the path of P_j at distance from A equal to ρ_j then a circle with center A and radius ρ_j cuts all paths of $P_{j-1}, P_{j-2}, \cdots, P_2$ in points which, joined to A, divide the angle into $(j-1)$ equal parts.



ON A PARTICULAR PLANE SECTION OF THE TORUS

JOSEPH D. E. KONHAUSER, HRB-Singer, Inc.

It is well known that the intersection of a plane through the center of a torus and tangent to the surface of the torus consists of two intersecting circles. However, it is not easy to find proofs of this fact in the literature. In this note we establish the assertion using elementary analytical geometry.

For example, let the torus be the surface generated by revolving the circle $(x-b)^2+z^2=a^2$, $y=0$, $a<b$, about the z-axis. The equation of the surface is then $(\sqrt{(x^2+y^2)}-b)^2+z^2=a^2$. Let the plane through the center of the torus and tangent to the surface of the torus be the plane containing the y-axis with trace $z=x\tan\theta$, where $\sin\theta=a/b$, in the xz-plane. Rotate coordinate axes so that the y-axis remains fixed and so that the tangent plane becomes the $x'y'$-plane. The equation of the torus in the new coordinate system is then

$$(\sqrt{((x'\cos\theta - z'\sin\theta)^2 + y'^2)} - b)^2 + (x'\sin\theta + z'\cos\theta)^2 = a^2.$$

To find the curve of intersection of the torus and plane we set $z'=0$ to obtain

$$x'^2 + y'^2 - 2b\sqrt{(x'^2\cos^2\theta + y'^2)} + b^2 = a^2.$$

Removing radicals and rearranging terms, we get

$$(x'^2 + y'^2)^2 - 2(b^2 - a^2)(x'^2 + y'^2) + (b^2 - a^2)^2 = 4a^2y'^2,$$

which may be written

$$[x'^2 + (y' - a)^2 - b^2][x'^2 + (y' + a)^2 - b^2] = 0.$$

Clearly, the intersection consists of two circles of radius b, with centers on the y'-axis (y-axis) at a distance $2a$ units apart.

Editor's Note: For a good drawing of the section discussed here, see Mathematical Games by Martin Gardner, *Scientific American*, 203 (Nov., 1960) 196.

SOME FIFTH DEGREE DIOPHANTINE EQUATIONS

W. R. UTZ, University of Missouri

The Diophantine equation

$$y^2 = 1 + x + x^2 + \cdots + x^n \tag{1}$$

is trivial in case $n=0$ or $n=1$. In case $n=2$, (1) is a Pell-type equation with a finite number of solutions $(x, y)=(0, \pm 1), (-1, \pm 1)$. For $n=3$ the equation was solved by Gerono [2] who discovered that the solutions are $(x, y)=(-1, 0)$, $(1, \pm 2)$, $(0, \pm 1)$, $(7, \pm 20)$ and A. A. Bennett [1] has given a very clever argument to show that the solutions of (1) for $n=4$ are $(x, y)=(-1, \pm 1)$, $(0, \pm 1)$, $(3, \pm 11)$. So far as I am aware, the cases $n \geqq 5$ have not been solved.

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Mordell [3] has given sufficient conditions that $y^2=P(x)$, $P(x)$ a polynomial of degree 3 or 4, shall have a finite number of solutions as is the case for $n=3$, 4 of (1). Equations with only a finite number of solutions are of special interest in the study of Diophantine equations which accounts, in part, for the interest in solutions of specific equations of the form $y^2=P(x)$, $P(x)$ a polynomial.

In case $n=3$, equation (1) becomes $y^2=(1+x)(1+x^2)$ which suggests the equation

$$y^2 = (1 + x)(1 + x^3) \tag{2}$$

which may be written

$$y^2 = (1 + x)^2(1 - x + x^2). \tag{3}$$

From (3) one has, if $x \neq -1$, $(y/1+x)^2=1-x+x^2$ which implies $y/1+x$ is an integer if x is an integer. Let $y/1+x=z$ to secure

$$z^2 = 1 - x + x^2. \tag{4}$$

Equation (4) may be written as

$$(2z - 2x + 1)(2z + 2x - 1) = 3 \tag{5}$$

which implies the system

$$2z - 2x + 1 = p$$
$$2z + 2x - 1 = q$$

where p, q are integers such that $pq=3$. Thus p, q are ± 1 or ± 3 and the four resulting systems give $(x, z)=(0, \pm 1)$, $(1, \pm 1)$ from which it is seen that all integral solutions of (2) are $(x, y)=(0, \pm 1)$, $(1, \pm 2)$ together with $(x, y) =(-1, 0)$.

This procedure suggests the generalization given by the following theorem, the proof of which is evident.

THEOREM. *If n, s are positive integers, $P(x)$ and $Q(x)$ are polynomials, then the equation*

$$y^n = [P(x)]^{s \cdot n}Q(x)$$

has as solutions $(r_i, 0)$, where r_i are integral solutions of $P(x)=0$, together with $(x, z[P(x)]^s)$ where (x, z) are solutions of $z^n=Q(x)$.

For example,

$$y^2 = (ax + b)^2(1 + x + x^2 + x^3)$$

will have as its solutions $(x_1, z_1(ax_1+b))$ where $(x_1, z_1)=(-1, 0)$, $(1, \pm 2)$, $(0, \pm 1)$, $(7, \pm 20)$ together with the solution $(-b/a, 0)$ in case a is a factor of b.

The fifth degree equation

$$y^2 = (ax^2 + bx + c)^2(dx + e) \tag{6}$$

may not have solutions (for example, if $d=5$, $e=3$ and $a^2+b^2+c^2 \neq 0$). To solve

(6), first discover the integral zeros, r_i, of ax^2+bx+c to secure the pairs $(r_i, 0)$. Then consider

$$z^2 = dx + e. \tag{7}$$

If $(d, e)=1$, then there is a solution of (7) if, and only if, e is a quadratic residue modulo d. In case d and e are not relatively prime, conditions for (7) to have a solution are also known [4].

Let d, e be integers for which (7) has a solution (x_1, z_1). Then for all integers n, $(x=x_1+2nz_1+n^2d,\ z=z_1+nd)$ is also a solution of (7); hence for such a pair of integers d, e the equation (6) has an infinite number of solutions unless the right hand side of (6) is identically a constant.

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1. A. A. Bennett, Problem, Amer. Math. Monthly, 33 (1926) 282–283.
2. E. Gerono, Nouv. Ann. Math., (2) vol. 16 (1877) pp. 230–234.
3. L. J. Mordell, On the integer solutions of $ey^2=ax^3+bx^2+cx+d$, Proc. London Math. Soc., (2), 21 (1921) 415–419.
4. Trygve Nagell, Introduction to number theory, Wiley, New York, 1951, Chap. 4.

FACTORIZATION OF $a^{2n}+a^n+1$

Charles T. Salkind, Polytechnic Institute of Brooklyn

We seek real polynomial factors of $a^{2n}+a^n+1$ with integer coefficients, where n is an integer, $n \geqq 1$.

If n is of the form $3k+1$, $k=0, 1, 2, \cdots$, there is a factor a^2+a+1. The case $n=1$ is trivial since the given expression becomes a^2+a+1 when $n=1$. For all other cases of $n=3k+1$, we note that both w and w^2 are zeros of the given trinomial, where $w=\frac{1}{2}(-1+i\sqrt{3})$, $i^2=-1$, for $(w)^{6k+2}+(w)^{3k+1}+1=w^2+w+1=0$ and $(w^2)^{6k+2}+(w^2)^{3k+1}+1=w+w^2+1=0$. The factor a^2+a+1 is obtained by taking the product $(a-w)(a-w^2)$.

Similarly, if n is of the form $3k+2$, $k=0, 1, 2, \cdots$ there is a factor a^2+a+1 since $(w)^{6k+4}+(w)^{3k+2}+1=w+w^2+1=0$ and $(w^2)^{6k+4}+(w^2)^{3k+2}+1=w^2+w+1=0$.

If n is of the form 2^r, $r=1, 2, \cdots$, there is, in addition, the factor a^2-a+1 since also $-w$ and $-w^2$ are zeros of the given trinomial, for $(-w)^{2^{r+1}}+(-w)^{2^r}+1$ equals either w^2+w+1 or $w+w^2+1$ and $(-w^2)^{2^{r+1}}+(-w^2)^{2^r}+1$ equals either $w+w^2+1$ or w^2+w+1.

If n is of the form 6^s, $s=1, 2, \cdots$ there is a factor $a^{6s}+a^{3s}+1$ and a factor $a^{6s}-a^{3s}+1$. The proof of this result is simple. If $s=2t$, $t=1, 2, \cdots$, then the factor $a^{6s}+a^{3s}+1$ can be further factored.

If n is of the form $6k+3$, $k=0, 1, 2, \cdots$ the given trinomial has no real polynomial factors with integer coefficients since (1) the given trinomial can not be factored as the difference of two squares and (2) neither w, w^2, $-w$, $-w^2$ are zeros of the trinomial.

(6), first discover the integral zeros, r_i, of ax^2+bx+c to secure the pairs $(r_i, 0)$. Then consider

$$z^2 = dx + e. \tag{7}$$

If $(d, e)=1$, then there is a solution of (7) if, and only if, e is a quadratic residue modulo d. In case d and e are not relatively prime, conditions for (7) to have a solution are also known [**4**].

Let d, e be integers for which (7) has a solution (x_1, z_1). Then for all integers n, $(x=x_1+2nz_1+n^2d,\ z=z_1+nd)$ is also a solution of (7); hence for such a pair of integers d, e the equation (6) has an infinite number of solutions unless the right hand side of (6) is identically a constant.

References

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A CONSTRUCTION OF REGULAR POLYGONS OF pq SIDES LEADING TO A GEOMETRIC PROOF OF $rp-sq=1$

PAUL B. JOHNSON, University of California, Los Angeles

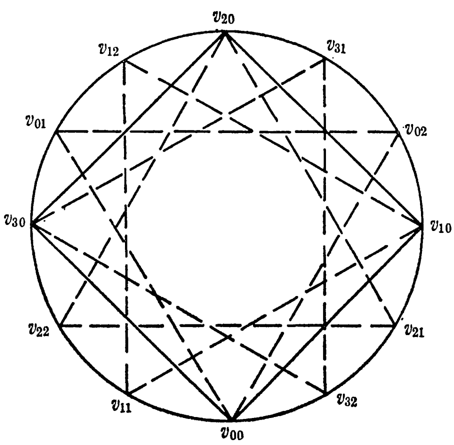
The construction of 15, 51, and 85 sided polygons is an interesting exercise when studying ruler and compass constructions. The method is based on the famous Greatest Common Divisor theorem that if p and q are relatively prime, there exist counting numbers r, s such that $rp-sq=1$. The proof of this theorem, while elementary, involves descent, and working back through a messy chain of equations. The spirit of the algebraic steps is quite different from that of geometric construction. Hence it is interesting in a geometry class to use a proof less dependent on an algebraic lemma. Indeed, a proof of the lemma falls out as a corollary.

THEOREM 1. *Let p and q be relatively prime. Suppose a way of inscribing regular p-gons and q-gons in a unit circle is known. The vertices of a regular pq-gon inscribed in the circle are the vertices found by*

(1) *Inscribing a regular q-gon*

(2) *Inscribing q different regular p-gons each with one vertex the same as one of the vertices of the q-gon.*

The figure illustrates the situation for $p=3$, $q=4$.



Proof. Label one vertex of the q-gon v_{00}. Then every vertex of a p-gon can be uniquely labeled v_{ij}: $1=0, \cdots, q-1$; $j=0, 1, \cdots, p-1$ as follows: v_{ij} is reached by moving positively (counter clockwise) from v_{00} along i edges of the q-gon and then back along j edges of the p-gon. We need two lemmas.

LEMMA 1. *The length of the arc joining v_{00} with v_{ij} is $2\pi(i/q-j/p)$.*

The proof is an interesting exercise.

LEMMA 2. *No two of the vertices of the p-gons coincide. For, if $v_{ij}=v_{kl}$, then the arc lengths from v_{00} being equal implies*

$$2\pi\left(\frac{i}{q}-\frac{j}{p}\right)=2\pi\left(\frac{k}{q}-\frac{l}{p}\right) \quad or \quad 2\pi\left(\frac{k}{q}-\frac{l}{p}\pm 1\right)$$

or

$$(i-k)p=(j-l)q \quad or \quad (j-l\pm p)q.$$

Because i and k are nonnegative and less than q, $|i-k|<q$. Since p and q are relatively prime and q divides $(i-k)p$, q divides $(i-k)$; hence $i-k=0$. $j-l=0$ follows and the vertices with different subscripts are different.

Of the pq vertices, one, say v_{rs}, must be the first after v_{00} in the positive direction. Let us rotate the whole configuration so v_{00} coincides with v_{rs}. The new configuration coincides with the old. For let v_{ij}^* be related to v_{rs} as v_{ij}^* was related to v_{00}. Then the arc from v_{00} to v_{ij}^* is $2\pi[(r+i)/q-(s+j)/p)]$. Hence $v_{ij}^*=v_{r+i,s+j}$. If $r+i$ or $s+j$ are greater than q or p respectively one goes completely around a polygon. Hence replace $r+i$ by $r+i-q$ or $s+j$ by $s+j-p$ respectively.

Since every vertex $v^*..$ is one of the old vertices $v..$, the entire set of vertices is invariant under a rotation through the arc v_{00} to v_{rs}.pq such rotations will return each vertex to its original position. Because v_{rs} was the next vertex after v_{00}, no rotation before the pqth will return a vertex to its original position. Hence the set of vertices v_{ij} are the vertices of a regular pq-gon.

COROLLARY 1. $1=rp-sq$.

Follows from equating two expressions for the arc from v_{00} to v_{rs}.

$$\frac{2\pi}{pq}=2\pi\left(\frac{r}{q}-\frac{s}{p}\right).$$

COROLLARY 2. *If one regular p-gon and one regular q-gon are inscribed in a circle with one vertex in common, a closest pair of remaining vertices are adjacent vertices in a regular pq-gon.*

The proof follows by choosing v_{00} in Theorem 1 as the vertex of the q-gon which is a member of the pair. The direction toward the neighboring vertex of the p-gon is the positive direction.

The method can be extended to prove the following more general theorem.

THEOREM 2. *If the greatest common divisor of p and q is d, then the construction of Theorem 1 yields a regular pq/d-gon. As a corollary, there exist integers r and s such that $rp-sq=d$.*

The proof is left as an exercise.

THE QUALIFYING EXAMINATION

RICHARD ROTH, University of Colorado

Drama in one act with 4 characters: The Grand Alpha, The Grand Beta, The Grand Omicron, and The Candidate.

As the curtain rises, Alpha, Beta, and Omicron are seated in a classroom of a large university and the Candidate comes in.

ALPHA: Who enters?

CANDIDATE: I am the Candidate.

ALPHA: State your purpose.

CANDIDATE: I come in pursuit of mathematical knowledge. I am prepared in the fundamental fields, Algebra, Analysis, Anti-derivation. You may question me.

ALPHA: The candidate will please define what is meant by a continuous denominator.

CANDIDATE: Consider the set of all doubly evocative singly homologous functions on the unit sphere. Introducing a continuous group structure in the usual way we may define the Skolem uniformity of automorphic cycles to be the theta relation on all sets of measure zero and the zeta function on left ideals whose valuation is Gaussian, uniformly on compacta. Then given any cardinal predicate, the continuous denominator is the corresponding normal quaternion for which the problem vanishes almost everywhere.

BETA: Could the candidate please give an example of a non-Skolem uniformity?

CANDIDATE: I believe the inversion of the reals under countable intersections is non-Skolem . . . at least almost everywhere.

BETA: That's correct. Now could you . . .

OMICRON: (Interrupting.) I wish to contradict. It isn't a non-Skolem uniformity since the third axiom concerning the density of the seventh roots of unity is not in fact satisfied.

BETA: Ah, yes, but you see, in my paper on toxic algebras . . . 1957 . . . *Journal of Refined Mathematics and Statistical Dynamics* of the University of Lompoc . . . I showed that the third axiom need not be satisfied if the basis is countably finite and the metric is Noetherian, hence . . .

ALPHA: (Interrupting.) Ahem, excuse me. The candidate will please prove the hokus-locus theorem on uniform trivialities.

CANDIDATE: By the Heine-Borel Theorem we reduce the Hamilton-Cayley equation to the canonical Cauchy-Riemann form. The Bolzano-Weierstrass property then shows that the Radon-Nikodym derivative satisfies the Jordan-Hölder relation. Hence by the Stone-Weierstrass approximation we can get the Schroeder-Bernstein map to be simply separable. The Lebesgue-Stieltjes integral then satisfies the Riemann-Roch result when extended by the Hahn-Banach method almost somewhere.

BETA: Please define a compact set.

CANDIDATE: A set is compact if every covering by open sets has a finite sub-opening. I mean every opening by finite sets has a compact subcovering. Er . . . rather, every compact by an open finite has a subcover. I mean a finact combine subopen if setcover set everying. That is, almost some of the time.

ALPHA: Leave that for a moment. Instead could you give us an example of a compact set.

CANDIDATE: Uh, you consider the real line and take any bounded subset, I mean closed subnet, er, I mean complete subsequence . . . bounded elements . . .

BETA: For example, is an interval compact?

CANDIDATE: Yes . . . er, I mean no . . . that is sometimes . . . almost everywhere? . . . if it is finite . . . or rational, I mean the irrationals—given a Dedekind cut—er, all the numbers less than $\sqrt{2}$ have a limit, that is . . .

OMICRON: Never mind. Look . . . is $\sqrt{2}$ rational or irrational?

CANDIDATE: It's rational . . . I mean it's not rational . . . $n^2=2m^2$ and all that . . . n less than m or I mean prime to 2 . . . they're all integers of course.

ALPHA: What do you mean by integers?

CANDIDATE: Well . . . there's Peano's postulates or axioms and there's this element 1 and $s(1)$ is 2 and $s(s(1))$ and so forth. I think almost everywhere and uh . . . yes.

BETA: We have a feeling that you are not quite sure of the material. For example, how much is 2 added to 2?

CANDIDATE: Well, we have a binary operation $+$, defined by induction and we let 2 denote . . . .

BETA: Never mind the proof . . . . Just tell us the ordinary name of the integer which results from adding the integer 2 to the integer 2.

CANDIDATE: Er . . . uh . . . that sounds familiar. I remember: 2 generates a prime ideal in a Dedekind domain, which is ramified when . . . .

ALPHA, BETA, and OMICRON in chorus: How much is 2 and 2? You learned it in the first grade?

CANDIDATE: Yes, oh yes . . . . I just can't think . . . I really know it . . . let me see . . . the first grade, you say. That's right . . . 2 plus 2 is . . . . Now first one plus one is two, one plus two is three, 8 times 8 is 65 . . . Stuff like that. 2 plus 2 is 2 plus 2 is 2 plus 2 is . . . .

ALPHA: That is quite enough. The examination is over. The candidate will write his name on the board while the committee deliberates on its decision. (The candidate, chalk in hand, stands facing the blackboard, writes a few letters on the board, erases them, looks blankly around the room as the curtain falls.)

(*Note:* This is a shortened version of a piece written in January, 1960 when the author was a graduate student at the University of California, Berkeley.)

ESCALATING INTEGRALS

GILBERT LABELLE, Université de Montréal

Application of Newton's process to the equation $\ln x - 1 = 0$ leads, in the $(k+1)$-th iteration, to

$$x_{k+1} = 2x_k - x_k \ln x_k = x_k - (x_k \ln x_k - x_k) = x_k - \int_0^{x_k} \ln x dx.$$

The process converges for $0 < x < e^2$. (For $x \geqq e^2$, the tangent line to $\ln x - 1$ would intersect the x-axis at $x = 0$ or to the left of $x = 0$.)

If we take $x_0 = \pi \in (0, e^2)$ we obtain the following formula

$$e = \pi - \int_0^{\pi} \ln x dx - \int_0^{\pi - \int_0^{\pi} \ln x dx} \ln x dx - \cdots$$

where the upper limit in each integral is equal to that section of the series that precedes the integral.

NUMBERS

GINSEY GURNEY, Fallston, Maryland

There are numbers I don't trust . . . take
One—too proud, too pointed, much too
Sure that all begins with him—and
Two, that sits cross-legged on the
Path and sneers as though he knows all
Secrets. Three's three levels claim such
Certainty that trinity is

Doctine—but I cannot trust it.
Four admits he might be cross, for
Ambiguities may move those
Arms to give salute too many
Ways. And five? Five clutches his wan
Hand and peers around his shoulder.
Who'd trust characters like these?

Six, no better, clasps the die that
Goes no further than his grasp, while
Seven wagers all he owns on
What he throws and leans to watch the
Play. Smug Eight, too self-content in
Knowing it's infinity set
Up right, keeps apart from all these

Others. Nine glares back at all of
Them, suspicious that he may be
Just a one who grew too swollen
From his pride. This surly tribe of
Arabics, their tenets folded,
Steal all reason blind. Don't count on
Them—these ciphers are impostors.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

On Teaching Mathematics. Edited by Bryan Thwaites. Pergamon Press, Oxford (and Macmillan, New York), 1961. xxiv+116 pp. $1.45.

Some Lessons in Mathematics, A Handbook on the Teaching of "Modern Mathematics." Edited by T. J. Fletcher. Cambridge University Press, Cambridge (and New York), 1964. xiii+367 pp. $2.95.

School Mathematics Project, Book T, Cambridge University Press, Cambridge (and New York), 1964, ix+296 pp. $3.50.

In the United States we are likely to be so busy with our own program for the reform of mathematical education that we are unaware of parallel studies being made in other countries. These three books offer a good view of what is currently happening in Great Britain. The first contains the proceedings of a ten-day conference held at Southampton in 1961 which helped set the stage for this activity. The second is the work of a "Writing Week" held in Leicester in September 1962 sponsored by the Association of Teachers of Mathematics. The third is a school textbook for ages 13–14 written by a curricular reform group under the chairmanship of Professor Thwaites. The three, together, enable us to see the aims of this British movement as well as the ways in which the details are being worked out. Since these differ considerably from those which are current in the United States, teachers in our country will find it profitable to examine these books carefully. We might just learn something!

In reading through these books I have been sharply reminded of my personal contacts with British mathematics. I "read" mathematics at Oxford in 1932–34 and spent more than half my time on applied topics, such as mechanics, which are not part of the syllabus in the United States. When I was a visitor in Cambridge in 1957–58 I gave a few lectures on mathematical reform in the United States, and it is probably no coincidence that my warmest reception was at the University of Southampton which is now the center of the British movement reported on in two of these books.

Traditionally British mathematics has been heavily oriented toward the practical side. Since Britain has relied so heavily on its technology for survival, it is natural that the schools were expected to train their "boys" in the mathematics which was used in industry. A few were allowed to study abstract mathematics, but even at Oxford and Cambridge applied mathematics forms half of the curriculum for the Honours B.A. In the United States we have tended to separate mathematics from its applications, so that university majors in mathematics frequently have only the most rudimentary ideas of its applications to physics, engineering, and industry. A corollary to this distinction is that British mathematics puts a premium on special methods ("tricks") to solve specific problems and emphasizes cleverness, whereas American mathematics tends to concentrate on structure and proof and emphasizes general principles. The differences between these books and SMSG (say) sharply reflect these two views of the nature of mathematical education.

The conference "On Teaching Mathematics" begins by demonstrating that the shortage of mathematicians and mathematics teachers in Britain is at least as critical as it is in the United States. (In 1957 I was assured that there was, indeed, a surplus of mathematicians in Britain!) The volume then proceeds to describe the kind of person whom industry would like to employ as a mathematician, and the remaining chapters are devoted to the means for training such people in the schools and universities. The last pages consist of a series of abstracts of lectures given at the conference by men from industry who describe the kinds of mathematics used in their concerns.

"Some Lessons in Mathematics" was written (or so it is claimed) in one week by a group of some 20 people. Its main purpose is to bring together materials from many portions of mathematics which are relatively modern, to illustrate their application to practical problems, and to show how they can be introduced in the schools. The very able introduction is completely in the spirit of thinking in the United States, but most of the remaining chapters are typically British.

The topics considered are: Binary Systems, Finite Arithmetics and Groups, Numerical Methods and Flow Charts, Sets—Logic—and Boolean Algebra, Relations and Graphs, Linear Programming, Patterns and Connections, Convexity, Geometry, Vectors, and Matrices. Each of these chapters is a mixture of intuition, application, and a light dash of theory. The applications are especially interesting and will repay careful study by Americans who know these subjects chiefly as chapters in pure mathematics. As an example of the richness of the approach, here are the topics included under Binary Systems: History, Binary Fractions, Binary Codes, and Delay Networks. Carefully constructed teaching lessons are included with an indication of the appropriate age at which the material can be absorbed. Here and there, directions are included for the home construction of imaginative teaching aids. There is an elaborate bibliography which is especially valuable because of its references to British works which are not well known in this country. In no sense is this a textbook; rather it is a source book which contains a host of excellent ideas not easily available to most teachers. It deserves a wide American circulation.

"Book T" (The notation, T, is unexplained but presumably means "transition") is a textbook for "boys" aged 13–14. The plans for the project call for Book 1 (age 11), Book 2 (age 12), . . . Book 5 (age 15); Book T is intended as a course for those entering the sequence in the third year after a previous traditional background.

It begins with three (too brief) chapters on numbers, sets, and inequalities. Presumably these topics will be covered in more detail in Books 1 and 2, and are here for purposes of transition. The main theme of the rest of the book is geometry. This is introduced by means of transformations in the plane: reflection, rotation, symmetry, translation, shearing, and enlargement. The philosophy is that of the Klein "Erlangen Program," but the approach is that geometry is a branch of physics. Coordinate and solid geometry and simple trigonometry are presented in due course.

Interspersed among the geometric chapters is material on approximate calculation, statistical graphs and charts, proportion, functions, rules of algebra, and binary arithmetic, but the main theme is geometry without axioms. Thus the influence of the applied thinking in the first two books is strongly apparent here; this is a book on physical geometry and related matters.

Teachers in the United States will find much of interest in this book for courses in intuitive geometry in the seventh and eighth grades. The design and printing are of an unusually high quality. American publishers have much to learn from the Cambridge University Press with regard to the manufacture of mathematics textbooks.

In my opinion neither the overly structured American approach nor the practical British approach is ideal for the teaching of school mathematics. Let us hope that the next generation of textbooks in both countries will arrive at a judicious combination of the two.

C. B. ALLENDOERFER, University of Washington

Computer Programming: A Mixed Language Approach. By Marvin L. Stein and William D. Munro. Academic Press, New York and London, 1964. 459 pp. $11.50.

A very thorough and precise presentation of the mechanics of digital computer programming in the classical textbook style make this book excellent for course instruction. It is filled with many well-planned exercises. These distinguished authors with many years of both practical programming and teaching experience reflect their capabilities in this most carefully written book.

It relies heavily upon a specific commercially available computer, the Control Data 1604, and a great part of the book is taken up with its description. This is the best way to teach programming. Reliance on an imaginary model or a generalization trying to cover most computers results in a very shallow and artificial presentation. The authors have been careful to describe many features of other computers which are frequently found and require significantly different programming methods. However, there are drawbacks to this approach. The student will find the book frustrating in parts unless he has access to the computer being described. Parts of the book become quickly dated as computers with new features and organization become available. This is unavoidable, and fortunately the student with the basic programming knowledge gained from this book will find transition to new and different computers a simple matter of adaption.

Number systems (33 pages), scaling, program organization, input-output, assemblers and compilers are well presented. The book is weak in flow-charting, checkout techniques, data organization and good documentation practices generally. The nomenclature is somewhat parochial, and a wider variety of input-output and storage devices is not discussed. But nevertheless, it is not meant that of the Klein "Erlangen Program," but the approach is that geometry is a branch of physics. Coordinate and solid geometry and simple trigonometry are presented in due course.

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D. B. PARKER, Control Data Corporation

Introduction to Statistical Method. By Sylvain Ehrenfeld and Sebastian Littauer. McGraw-Hill, New York, 1964. vii+533 pp. $9.75.

This book will serve well as an introductory text to mathematical statistics for undergraduate engineering and science students. The authors are to be credited for their approach, which emphasizes the power of statistics and probability in a decision process. Chapter Six, Decision Making, is a lucid discussion of decision under uncertainty. The development of this topic at the level of this text has been long overdue.

The text has several weak points, any of which can be overcome by a good instructor. Much of the text is spent in motivating the student; the effort in this behalf should have been devoted to shoring up the mathematical arguments which are often "beyond the scope of the present work." Many important points and worthwhile results lose their significance because they are presented as examples rather than in the main body of the text.

The second chapter is devoted to introductory probability notions. This chapter is too brief. For example, Bayes Rule receives only passing mention in an exercise. The concepts introduced in Chapter Three, Distribution Functions, may be blurred to the student because of undue formalism in notation. Functions of random variables are handled poorly in a probabilistic fashion rather than through the more powerful change of variable technique; many later arguments suffer without this technique. A good and thorough discussion of mathematical expectation is given in Chapter Four, Properties of Distributions. Various distribution functions are introduced in Chapter Five. The importance of each function and the interrelationships of a number of functions are clearly stated. To the credit of the authors, the notion of the regret function introduced in Chapter Six is carried to Chapter Seven on hypothesis testing. The power function and its relation to types of possible alternatives in hypothesis testing is clearly explained. Estimation procedures are developed from the pragmatic viewpoint of desirable properties in the estimators. The chapter on analysis of variance contains a good discussion of power of F tests and of linear contrasts. The authors could have done with a less hurried discussion of the basic notions of ANOVA and included more numerical examples. Regression theory is well handled with a great number of meaningful statistical tests clearly developed. The first and last chapters confuse rather than explain and could well have been omitted.

In general, the exercises in the text strike a good balance between theoretical and applied problems and encompass a fair range of difficulty. There is not an answer book to the exercises.

JAMES ASSING, Booz, Allen Applied Research, Inc., Los Angeles, California to be all things to all people, and as a good, solid basic text on programming, it is excellent. These limitations can and should be made up for by auxiliary texts, current publications and a well-informed instructor.

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JAMES ASSING, Booz, Allen Applied Research, Inc., Los Angeles, California

Calculus and Analytic Geometry. By Melcher P. Fobes and Ruth B. Smyth. Prentice-Hall, Englewood Cliffs, N. J., 1963. Vol. 1, 660 pp. \$8.50. Vol. 2, 450 pp. \$6.95.

We have here a pair of well-printed and bound books, one for each of the first two years of college mathematics. The only prerequisites are the usual high school courses in algebra and geometry. The basic concepts of algebra and trigonometry are presented and there is a full chapter on determinants. The exercises are plentiful and graded in difficulty with the answers to half of them at the back of the book. The figures are also plentiful and well drawn, especially those representing solid objects. Each volume has an excellent index.

The exposition is exceptionally full. This in itself may be a virtue in a mathematics text. For the precision of mathematical concepts and the logic of the mathematician often lead him to pack exactly what he has to say into "a few well-chosen words." This is fine in an article for experts, but in a textbook it leaves the ordinary student somewhat stunned. The authors of our text lead the student along slowly with an easy conversational style and he gradually gets the idea and remains undisturbed.

It may be objected that this style of exposition is often lacking in rigor, and that rigor in a mathematics text is highly desirable or even necessary. But the student may find it excessive. Thoughtful students in high school geometry often ask, "Why must we prove this theorem? It is perfectly obvious in the first place. The proof is nothing but hair splitting." Bertrand Russell has said, "Hair splitting is a degree of refinement in reasoning beyond that of which the speaker is capable." For textbook writers we may amend this to read—Hair splitting is a degree of refinement in reasoning beyond that of which the student is capable. Our authors have found an ingenious way around the difficulty. They first present a simple intuitive discussion, and then after several pages of application and illustration they develop a new statement embodying the desired rigor.

Each chapter is followed by a study outline and also by a collection of challenging problems and suggestions under the heading, "Things to think about."

These books are unreservedly recommended for the combined course. Of course I like to see historical notes in a mathematics text, but perhaps it is all right to leave something for the teacher to do.

C. J. COE, University of Michigan

Basic Statistics: A Primer for the Biomedical Sciences. By Olive Jean Dunn. Wiley, New York, 1964. 184 pp. \$5.50.

This book is designed for students in the medical fields, such as medical students, dental students, student nurses, etc. It requires a mathematical background extending through arithmetic and enough algebra to use symbols. In general, it is clearly written. The style is pleasant and the quality of explanation is good. About one-fourth of the book is devoted to descriptive statistics; the other three-quarters is devoted to statistical inference, and covers such topics as confidence intervals for means and differences between means, tests of

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Generally, the topics included are well-chosen, but it can be objected that the chapter on variances would be better left out. The reason is that the normal theory procedures presented are really of very questionable validity unless the data actually come from normal distributions. The same objection does *not* apply to the other procedures in the book. Tests about means, regression coefficients, etc., although derived under normal theory, are quite robust when applied to the kind of data one actually meets in practice. In fairness to the author, it should be said that any elementary book is likely to include such a chapter subject to the same objection.

Were the book somewhat longer (it is less than 200 pages), its readers might be gratified to find presentations of the analysis of variance (in its simplest cases) and bioassay. However, in general, the most important topics are the ones that have been chosen, and they have been well-presented. Most chapters close with a number of exercises, which are generally interesting; answers to some of them appear in the back of the book. To aid the student in monitoring the appearance of new ideas, a glossary is given at the end of each chapter. Finally, there is a satisfactory index and a brief collection of useful tables.

The book is notable for its correctness, as well as for its readable style. It is a welcome addition to the small set of usable books for teaching medical students.

L. E. MOSES, Stanford University

Introduction to Analog Computation. By Robert J. Ashley. Wiley, New York, 1963. 294 pp. $8.75.

A person interested in learning about the basis of analog computing will find this book a useful starting point. Emphasizing the role of the computer in solving problems, the author has written with clarity and has covered virtually all of the essential phases of basic analog computing, including iterative computation. I would particularly commend the author for his fine discussion of programming, as well as time and amplitude scaling. In addition, he describes how the computer performs its operations, e.g., addition, multiplication.

However, the author does falter badly on one occasion. After successfully highlighting, by example, most of the differences of analog computer, digital computer, and analytic solutions of a simple pendulum, the author reaches some erroneous conclusions (page 25). His conclusions should be corrected to read:

1) An analog computer *can* solve problems in which variations in parameters exceed one part in 10^3 (by the method of variable scaling).
2) The accuracy of the final solution is *not* always reduced more by leaving terms out of the equations than by the errors inherent in the solution. (It all depends upon the nature of the terms being omitted.)

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I also believe that the author could have improved his book with a discussion of computer errors and their effect on problem solution error. Even a novice should know that computers are fallible devices and he should have some idea of how to cope with their errors.

Yet, for all of these shortcomings, there is no elementary analog computer book better than this one.

LEON LEVINE, Scientific Data Systems

A Course of Mathematical Analysis. By A. F. Bermant. Translated from Russian by D. E. Brown, and edited by I. N. Sneddon. Pergamon Press, Macmillan, New York, 1963. Vol. I, xiv+493 pp. $10.00, Vol. II, xi+374 pp. $9.00.

This work in two volumes consists of fifteen chapters, with titles as follows: Vol. I: 1. Functions 2. Limits 3. Derivatives and differentials. The differential calculus 4. The investigation of functions and curves 5. The definite integral 6. The indefinite integral. The integral calculus 7. Methods of evaluating definite integrals. Improper integrals 8. Applications of the integral 9. Series. Vol. II: 10. Functions of several variables. Differential calculus 11. Applications of the differential calculus 12. Multiple integrals and iterated integration. 13. Line and surface integrals 14. Differential equations 15. Trigonometric series.

The preface makes clear that the book is intended primarily for the training of engineers and scientists. The following quotation from the preface conveys the intent: "The creative work of the engineer, both in science and industry, requires genuine mathematical knowledge, and not merely the ability to carry out formal mathematical operations. . . . The correct approach to this must lie in the presentation of a course in accordance with the basic scheme: practice—basic concepts of analysis—their properties (theory)—methods of working—methods of application—practice . . ."

The entire work is surprisingly close in organization, choice of subject matter, and general tone, to American texts of the pre-World War II period. The book seems distinctly unmodern, and it offers nothing particularly novel or charming in the exposition of familiar material. Proofs which require a fundamental penetration into the nature and consequences of the completeness of the real number system are avoided, but the necessary fundamental theorems are stated without proof.

There are illustrative examples, but no exercises or problems for the student.

The Introduction contains several quotations from F. Engels and Lenin. There are about four pages of comments of an historical nature, with special prominence for "Great Russian mathematicians L. P. Euler, N. I. Lobachevskii, P. L. Chebyshev" and "leading Russian applied mathematicians N. E. Zhukovskii, S. A. Chaplygin, A. N. Krylov." Ostrogradskii, Lyapunov, and A. A. Markov were also mentioned. To be sure, Euler is acknowledged to be Swiss by

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The following interesting statements are noted: "Lobachevskii was the first clearly to show the physical origins of the axioms of geometry, thus refuting the teaching of the German idealist philosopher Kant regarding their *a priori* nature" (vol. 1, p. 10). Concerning Newton: "His work helped to overthrow the medieval scholastics and was an invaluable asset in developing a genuinely scientific materialistic world-picture" (vol. 1, p. 137). "The subtle difference between continuous and differentiable functions was first remarked by the great Russian mathematician Lobachevskii" (vol. 1, p. 172).

In general the exposition is good. There are some lapses from accuracy in definitions and assertions. Examples: On pp. 453–454 of volume 1, in the theorem on term-by-term differentiation of a series of functions, it is not mentioned in the theorem, but tacitly assumed in the proof, that the original series is supposed to be convergent on an interval. On p. 10 of volume 2, the definitions of simply and doubly connected domains will not do for what the author wants. On pp. 18–19 of volume 2 three theorems are stated about a function which is assumed to be continuous in a closed domain. For the first and third theorem the domain needs to be bounded as well as closed. For the second theorem the domain needs to be connected. In none of these cases is the required condition mentioned. The proofs are not given.

A. E. TAYLOR, University of California, Los Angeles

Calculus. Problems and Solutions. By A. Ginzburg. Holden-Day, San Francisco, 1963. x+455 pp. $7.75.

This book is concerned with differential and integral calculus for functions of a single variable, and with sequences and infinite series. There are applications of differential calculus to rates, to the study of curves, to inequalities and approximation, and to curvature and evolutes. The applications of definite integrals range over the usual geometrical and physical topics which have been conventional in American calculus texts for many decades. The entire book seems to be very much along the lines of American calculus books in the selection of material and in point of view. On the whole, the problems are of the type that demands skill in standard techniques.

There are 1284 problems. A good many of them are solved immediately following their presentation. For instance, about one third of the 49 problems on maxima and minima are so solved. For the remaining problems, either solutions, hints, or answers are presented separately in the later portion of the book. This later portion occupies 178 pages out of a total of 450.

The book is not intended to serve as an exposition of calculus. Concepts and definitions are presented briefly. Basic rules, formulas, and theorems are given without proof. For example, Taylor's formula is given with two forms of the remainder: Lagrange's form and Cauchy's form. In some cases theorems are birth. In Russian style, the patronymic Pavlovich has been added to Euler's name. (His father's name was Paul.) Zhukovskii, called the founder of the science of aviation, is credited with the mathematical discovery of the possibility of "looping the loop" and other "figures of higher pilotage."

The following interesting statements are noted: "Lobachevskii was the first clearly to show the physical origins of the axioms of geometry, thus refuting the teaching of the German idealist philosopher Kant regarding their *a priori* nature" (vol. 1, p. 10). Concerning Newton: "His work helped to overthrow the medieval scholastics and was an invaluable asset in developing a genuinely scientific materialistic world-picture" (vol. 1, p. 137). "The subtle difference between continuous and differentiable functions was first remarked by the great Russian mathematician Lobachevskii" (vol. 1, p. 172).

In general the exposition is good. There are some lapses from accuracy in definitions and assertions. Examples: On pp. 453–454 of volume 1, in the theorem on term-by-term differentiation of a series of functions, it is not mentioned in the theorem, but tacitly assumed in the proof, that the original series is supposed to be convergent on an interval. On p. 10 of volume 2, the definitions of simply and doubly connected domains will not do for what the author wants. On pp. 18–19 of volume 2 three theorems are stated about a function which is assumed to be continuous in a closed domain. For the first and third theorem the domain needs to be bounded as well as closed. For the second theorem the domain needs to be connected. In none of these cases is the required condition mentioned. The proofs are not given.

A. E. TAYLOR, University of California, Los Angeles

Calculus. Problems and Solutions. By A. Ginzburg. Holden-Day, San Francisco, 1963. x+455 pp. $7.75.

This book is concerned with differential and integral calculus for functions of a single variable, and with sequences and infinite series. There are applications of differential calculus to rates, to the study of curves, to inequalities and approximation, and to curvature and evolutes. The applications of definite integrals range over the usual geometrical and physical topics which have been conventional in American calculus texts for many decades. The entire book seems to be very much along the lines of American calculus books in the selection of material and in point of view. On the whole, the problems are of the type that demands skill in standard techniques.

There are 1284 problems. A good many of them are solved immediately following their presentation. For instance, about one third of the 49 problems on maxima and minima are so solved. For the remaining problems, either solutions, hints, or answers are presented separately in the later portion of the book. This later portion occupies 178 pages out of a total of 450.

The book is not intended to serve as an exposition of calculus. Concepts and definitions are presented briefly. Basic rules, formulas, and theorems are given without proof. For example, Taylor's formula is given with two forms of the remainder: Lagrange's form and Cauchy's form. In some cases theorems are stated with full accuracy of hypothesis and conclusion (e.g., Rolle's theorem). In other cases the statements are more informal: nothing definite is said about differentiability conditions for Taylor's formula with remainder, where the remainder involves an n-th derivative. I believe that greater care should have been taken in stating some of the theorems on uniform convergence. For instance, here is the statement (page 242) of the assertion on term-by-term differentiation of an infinite series: "If all functions in the convergent series $u_1(x) + u_2(x) + \cdots$ are differentiable and if the series of derivatives $u_1'(x) + u_2'(x) + \cdots$ converges uniformly in a certain domain, then in that domain

$$[u_1(x) + u_2(x) + \cdots]' = u_1'(x) + u_2'(x) + \cdots."$$

I do not know of any proof of this assertion without imposing further conditions. For example, if the domain in question is a closed interval, it is sufficient to add the requirement that all the derivatives u_n' be continuous on the interval.

The book can be quite useful to an industrious student, and also to teachers of calculus who seek to enrich their repertory of problems.

The level of rigor is satisfactory, in general, at least where pure analysis is involved. In discussing the applications of the definite integral, however, the author's attempt to distinguish between heuristic arguments and rigorous proof is not very useful or successful, it seems to me. He begins with a quick heuristic derivation of the formula for obtaining the volume of a solid of revolution. He then says: "It must be understood that this heuristic argument is lacking in rigor and may lead to errors if applied indiscriminately." The subsequent brief discussion of how to avoid errors is inadequate. The final sentences of this discussion are: "To conclude, we remark that all formulas of this chapter can of course be given rigorous proof. Nevertheless, the heuristic reasoning, when performed with sufficient care, is very useful in practical problems." There are two separate matters here, neither of which is properly within the scope of this book to discuss in detail. The book does not recognize the distinction between the two matters, and it does nothing effective to guide heuristic reasoning to correct conclusions. One issue is the *definition* of such things as area, arclength, volume, centroid, moment of inertia, and so on. Sometimes, but not in all cases, the definition leads directly to an integral. The other issue is that of proving that certain kinds of limits of sums can be expressed as integrals, by some version or special case of Duhamel's principle.

I will conclude this review with some comments on matters of detail. The criticisms here have mainly to do with blemishes which are inadvertent, and not characteristic of the general quality of the book.

The radius of curvature R is introduced by means of an osculating circle, and curvature is defined as $1/R$. Thus there is no indication of the meaning of curvature as rate of turning of the tangent line, with respect to arclength along the curve.

The notation $\overline{f(x)}$ is used for the average value of $f(x)$ on a specified interval (page 180); this seems a poor usage.

It would have been better if each diagram had been placed on the same page with the material which refers to it; this is not always the case.

On page 214 the formula $P=2\pi\int_a^b ydL$ (for area of a surface of revolution) violates the usual rule about limits of integration, in that a and b are not values of L, but of x.

The theorem stated at the bottom of page 237, about a divergent series remaining divergent when each term is multiplied by a constant, is clearly false if the factor is zero.

The material on multiplication of two infinite series, especially the definition on page 238, is open to objection.

A formula is given for the radius of convergence of a power series "when the difference between the exponents in consecutive terms is k." But it is not made clear that a_n in the formula is the coefficient of x^{kn}, not x^n.

The author does not express himself properly in saying (pages 240 and 441) that the difference $\sum_1^\infty 1/n-\sum_1^\infty 1/(n-1)$ converges, while the difference $\sum_1^\infty 1/n-\sum_1^\infty 1/(2n)$ diverges.

A. E. TAYLOR, University of California, Los Angeles

Truth-functional Logic. By J. A. Faris. The Free Press, New York, 1963. 122 pp. \$1.25.

This excellent little book is one of the *Monographs in Modern Logic* edited by G. B. Keene. It is clear, well written, and careful. Its function is to provide an introduction to the techniques of sentential logic, and it performs this function in a simple, unpretentious manner.

The first two chapters introduce the notion of truth-functors and a notation for them. Given a standard order in which the possiblities of truth (1) and falsity (0) of propositions, or pairs of propositions, are to be listed, the notation for a truth functor renders its meaning clear; given the usual order ($p=1$, $q=1$; $p=1$, $q=0$; $p=0$, $q=1$; $p=0$, $q=0$), the truth functor T_{1000} for conjunction *shows* that only in the first of these cases is $T_{1000}(p, q)$ to have the value 1. When these truth functors have served their purpose, however, they are dropped; the main part of the book is written in standard ($.$, $\vee$, $\supset$, $\equiv$, $\sim$) notation.

The third chapter introduces the truth table method and applies it to the evaluation of arguments; the fourth chapter does the same thing for the deductive method. (The system presented is that of Copi's *Symbolic Logic.*)

The fifth chapter consists of two parts: the introduction of conjunctive normal forms as a means of evaluating arguments, and as a means of establishing a completeness theorem; and a discussion of the "applicability and limitations" of truth-functional logic. This section is not up to the others; it is here for the first time that the author introduces the word "implies" in relation to conditional statements, and the section is largely devoted to the attempt to make sense out of this strange interpretation of the horseshoe.

There is a list of suggested readings at the end of each chapter; but there are no exercises. There is a convenient list of definitions and abbreviations, and an index.

HENRY KYBURG, Wayne State University

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HENRY KYBURG, Wayne State University

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

PROPOSALS

586. *Proposed by Maxey Brooke, Sweeny, Texas.*

"Jim Clark told me that he saw a flying saucer." I told Ford.

"You can't believe a word Clark says." Ford answered.

"That's peculiar," I replied with my usual degree of truthfulness, "Clark said just the opposite about you."

What is the probability that Clark saw a flying saucer?

587. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove the following inequality

$$\left(\frac{\theta + \sin\theta}{\pi}\right)^2 + \cos^4 \frac{1}{2}\theta < 1, \qquad (-\pi < \theta < +\pi).$$

588. *Proposed by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.*

Show that the operators $(D-1)^n \times (D-1)$ and $x(D-1)^{n+1} + n(D-1)^n$ are equivalent for $n = 1, 2, 3, \cdots$, where $D \equiv d/dx$.

589. *Proposed by Charles W. Trigg, San Diego, California.*

a) Show that four upright cups all can be inverted by turning over three at a time in exactly $2n$ moves, $n = 2, 3, \cdots$.

b) The four upright cups can be inverted by turning over two at a time in exactly n moves, $n = 2, 3, \cdots$.

c) One inverted and three upright cups cannot all be inverted by turning over two at a time.

590. *Proposed by R. J. Cormier, University of Missouri.*

Show that the following set of simultaneous equations has no solution in distinct positive integers:

$$a^3 + b^3 = c^3 + d^3$$
$$a + b = c + d.$$

591. *Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada.*

Show that the nth Fibonacci number (defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$) satisfies the inequality

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} < F_n < \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \quad \text{for } n \geqq 3.$$

592. *Proposed by J. S. Vigder, Defence Research Board of Canada, Ottawa, Canada.*

Determine the values of n for which $\sum_{k=1}^{n} k^5$ is a perfect square.

SOLUTIONS

LATE SOLUTIONS

Maxey Brooke, Sweeny, Texas: **558**; *Martin Gorfinkel, Stanford Research Institute*: **558**; *Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania*: **559**; *Douglas Lind, University of Virginia*: **561**; *G. L. N. Rao, Jamshedpur Cooperative College, Jamshedpur, Bihar, India:* **559.**

Heir-Brained

565. [November, 1964] *Proposed by Maxey Brooke, Sweeny, Texas.*

Henry, Hyram and Hyman inherited a circular farm. It is not enough to divide it into three equal areas. In order that they can share equally the fencing costs, the three portions must also have equal perimeters. Can you help the heirs?

Solution by Charles W. Trigg, San Diego, California.

Assuming a flat terrain or one uniformly distorted from the level, all that is necessary is to divide the circular periphery into three equal parts and then to join each division point to the center by uniform fences such as could be gotten by rotating one of the joins 120° either way about the center. Of course, the joins should not intersect.

One wonders how a circular farm ever happened, and also about the disposition of the surrounding country.

Also solved by Philip Fung, Fenn College, Ohio; Dennis P. Geller, Harpur College, New York; Harry W. Hickey, Arlington, Virginia; Larry Hoehn, Perryville, Missouri; J. A. H. Hunter, Toronto, Ontario, Canada; Richard A. Jacobson, South Dakota State University; Vida Katkin, Harpur College, New York; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Douglas Lind, University of Virginia; and the proposer.

Geller pointed out that a solution in terms of two parallel secant lines does not exist.

A Square-Free Condition

566. [November, 1964] *Proposed by Martin J. Cohen, Beverly Hills, California.*

Prove that n is a square-free integer (i.e., the product of distinct primes) if and only if

$$\sum_{d|n} \phi(d)\sigma(d^{k-1}) = n^k$$

for all integers $k \geqq 2$.

Solution by L. Carlitz, Duke University.

Since

$$1 + \phi(p)\sigma(p^{k-1}) = p^k$$

it follows at once (since both sides are factorable functions of n) that

$$(*) \qquad \sum_{d|n} \phi(d)\sigma(d^{k-1}) = n^k$$

for square-free n and all $k=1, 2, 3, \cdots$. We now show that, conversely, if (*) holds for one value of $k>2$, then n must be square free. Indeed, since

$$\begin{aligned}\sum_{d|p^r} \phi(d)\sigma(d^{k-1}) &= 1 + (p^k - 1) + p(p^{2k-1} - 1) + \cdots + p^{r-1}(p^{rk-r+1} - 1)\\ &= (1 + p^k + \cdots + p^{rk}) - (1 + p + \cdots + p^{r-1})\\ &= (p^k + \cdots + p^{rk}) - (p + \cdots + p^{r-1})\\ &= p^{rk} + \sum_{j=1}^{r-1} (p^{jk} - p^j) > p^{rk}\end{aligned}$$

for $k>1$, $r>1$, it follows that the left member of (*) is greater than the right member whenever $k>1$ and n is not square free.

Also solved by John A. Burslem, St. Louis University; R. L. Duncan, Lock Haven State College, Pennsylvania; Henry W. Gould, West Virginia University; A. M. Vaidya and A. A. Gioia (jointly), Texas Technological College; and the proposer.

Analog of Morley's Theorem

567. [November, 1964] *Proposed by L. Carlitz, Duke University.*

Points A_1, A_2 are marked on the side BC of the triangle ABC so that $BA_1 = A_1A_2 = A_2C$, similarly B_1, B_2 on CA and C_1, C_2 on AB. Let A' be the point of intersection of BB_1 and CC_2, B' of CC_1 and AA_2, C' of AA_1 and BB_2. How is the triangle $A'B'C'$ related to ABC?

Solution by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.

Consider the following more general problem. Let each side of triangle ABC be divided into n equal parts. Let A_1, B_1, C_1, A_2, B_2, C_2 be such that $nAC_1 = nBC_2 = AB$, $nBA_1 = nCA_2 = BC$, $nCB_1 = nAB_2 = AC$. Let A', B', C' be the points of intersection of BB_1 and CC_2, CC_1 and AA_2, AA_1 and BB_2, respectively.

Select axes so that A, B, C have coordinates $(0, 0)$, $(a, 0)$, (b, c), respectively, then the coordinates of A', B', C' are easily determined to be $((n-1)(a+b)/(2n-1),\ (n-1)c/(2n-1))$, $((n-1)b+a/(2n-1),\ (n-1)c/(2n-1))$, $((n-1)a+b/(2n-1),\ c/(2n-1))$, respectively.

Triangles ABC and $A'B'C'$ have parallel sides and are, therefore, similar. The altitude on side $A'B'$ from C' is $(n-2)c/(2n-1)$, so the perimeter of triangle $A'B'C'$ is $(n-2)/(2n-1)$ times that of triangle ABC, and their areas are in the ratio $(n-2)^2/(2n-1)^2$.

Let P, Q, R be the points of intersection of BB_2 and CC_1, AA_1 and CC_2, AA_2 and BB_1, respectively; then P, Q, R have coordinates $((a+b)/(n+1), c/(n+1))$, $((n-1)a+b/(n+1), c/(n+1))$, $((n-1)b+a/(n+1), (n-1)c/(n+1))$, respectively.

The sides of triangles PQR are parallel to the sides of triangle ABC, so the triangles ABC, $A'B'C'$ and PQR are similar.

The altitude on side PQ from R is $(n-2)c/(n+1)$, so statements may be made about the ratios of the perimeters and areas of the three triangles.

Any two of the three triangles are perspective from each of the vertices of the other two; all three triangles have the same centroid; and so on.

Also solved by Maxey Brooke, Sweeny, Texas; John A. Burslem, St. Louis University; Sidney Spital, California State Polytechnic College; Charles W. Trigg, San Diego, California; J. S. Vigder, Defence Research Board of Canada; and the proposer.

Divisible Sums

568. [November, 1964] *Proposed by C. W. Trigg, San Diego, California.*

What is the nature of n if $\sum_{k=1}^{n} k^6$ is divisible by $\sum_{k=1}^{n} k^2$?

Solution by Raymond E. Whitney, Lock Haven State College, Pennsylvania. From the well-known formulae

$$\sum_1^N K^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_1^N K^6 = \frac{N(N+1)(2N+1)}{42}(3N^4+6N^3-3N+1),$$

we obtain

$$\sum_1^N K^2 \,\Big|\, \sum_1^N K^6 \text{ iff } 7 \,\Big|\, 3N^4+6N^3-3N+1.$$

Thus

$$3N^4+6N^3-3N+1 \equiv 0 \pmod 7$$
$$3N^4+6N^3-3N-6 \equiv 0 \pmod 7$$
$$(N-1)(N+2)(N^2+N+1) \equiv 0 \pmod 7$$
$$(N-1)(N+2)(N+3)(N-2) \equiv 0 \pmod 7.$$

Thus

$$\sum_1^N K^2 \,\Big|\, \sum_1^N K^6 \text{ iff } N \equiv 1, 2, 4, 5 \pmod 7.^*$$

* (This follows since $(I/(7), +, \cdot)$ is a field and has no zero divisors other than 0.)

Also solved by Arlo D. Anderson, U. S. Naval Research Laboratory, Washington, D. C.; Joseph Arkin, Spring Valley, New York; John A. Burslem, St. Louis University; L. Carlitz, Duke University; J. E. Connett, University of Missouri; Romae J. Cormier, Northern Illinois University; Harry M. Gehman, SUNY at Buffalo, New York; Henry W. Gould, West Virginia University; Roy H. Hines, Jr., Concord, Massachusetts; Stephen Hoffman, Trinity College, Connecticut; J. A. H. Hunter, Toronto, Ontario, Canada; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Sidney Kravitz, Dover, New Jersey; Herbert R. Leifer, Pittsburgh, Pennsylvania; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania; Joseph E. Mueller, UICSM, Urbana, Illinois; F. D. Parker, SUNY at Buffalo, New York; David L. Silverman, Beverly Hills, California; Arnold Singer, Institute of Naval Studies, Cambridge, Massachusetts; J. S. Vigder, Defence Research Board of Canada; Charles Ziegenfus, Madison College, Virginia; and the proposer.

Equal Zones

569. [November, 1964] *Proposed by Frank Dapkus, Seton Hall University.*

Is it true that of all the surfaces of revolution only the sphere and circular cylinder have the property that the areas of zones of equal thickness are equal?

Solution by R. A. Jacobson, South Dakota State University.

Let y be a function of x over the domain $[a, b]$. If surface area is to be equal for all zones of equal thickness, then

$$\int_{c-t}^{c} y\sqrt{(1 + (y')^2)}dx = \int_{c}^{c+t} y\sqrt{(1 + (y')^2)}dx \tag{1}$$

for all c, t such that $a \leqq c-t \leqq c+t \leqq b$. Differentiating (1) with respect to t, and letting $u=c-t$, $v=c+t$; we find that

$$y(u)\sqrt{(1 + [y'(u)]^2)} = y(v)\sqrt{(1 + [y'(v)]^2)},$$

$a \leqq u \leqq v \leqq b$. Thus $y\sqrt{(1+(y')^2)}$ must be constant throughout $[a, b]$. Letting $y\sqrt{(1+(y')^2)}=m$, we have

$$y^2(1 + (y')^2) = m^2, \qquad (y')^2 = \frac{m^2 - y^2}{y^2}, \tag{2}$$

$$\frac{y\,dy}{\sqrt{(m^2 - y^2)}} = \pm\, dx. \tag{3}$$

Assuming $m^2-y^2 \neq 0$, the solution for (3) is

$$-\sqrt{(m^2 - y^2)} = \pm\, x + A. \tag{4}$$

Substituting (4) into (2), we find that $A=0$ and hence $x^2+y^2=m^2$ satisfies (2).

If $m^2-y^2=0$, we have the singular solution $y = \pm m$. Since the only solutions to (2) are the circle and horizontal line, the question is answered in the affirmative.

Also solved by Harry W. Hickey, Arlington, Virginia; and the proposer.

A Bank Shot

570. [November, 1964] *Proposed by Leon Bankoff, Los Angeles, California.*

A billiard ball is placed at a point P on a circular billiard table with center O and radius $R=1$, and is hit so that it returns to its starting point after rebounding from points A and B on the circumference. If PD is the altitude to side AB of the triangle PAB determine the location of P so that O divides PD in the Golden Ratio $(\sqrt{5}-1)/2$. Neglect friction and spin of the ball, and assume perfect elasticity of the cushions.

Solution by Charles W. Trigg, San Diego, California.

The Golden Ratio is $\phi=(\sqrt{5}+1)/2 \doteq 1.618$. If O falls on PD, then PD is the perpendicular bisector of AB. Let $OD=x$. If $PO/OD=\phi$ and $OA=1$, then from the right triangles AOD and APD,

$$AD = \sqrt{(1-x^2)} \quad \text{and} \quad AP = \sqrt{(1-x^2+(x+\phi x)^2)} = \sqrt{(1+(2\phi+\phi^2)x^2)}.$$

By the law of reflection, AO bisects angle PAD, so

$$\frac{\phi x}{x} = \frac{\sqrt{(1+(2\phi+\phi^2)x^2)}}{\sqrt{(1-x^2)}}.$$

Simplifying and solving:

$$x = \sqrt{\left(\frac{\phi-1}{2\phi}\right)} = \frac{\sqrt{5}-1}{2\sqrt{2}}.$$

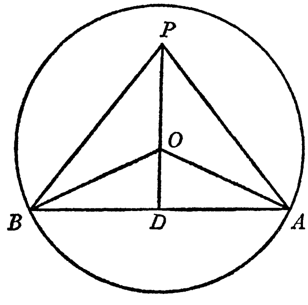
Whereupon,

$$OP = \left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{\sqrt{5}-1}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \doteq 0.707,$$

and P must lie on a circle with center O and radius $1/\sqrt{2}$.

$$PD > OD \text{ of necessity, and } \frac{PD}{OD} = \frac{\sqrt{5}+1}{2} = \frac{2}{\sqrt{5}-1},$$

so the relationship given in the problem leads to the same result.



Also solved by Brother T. Brendan, St. Mary's College, California; John A. Burslem, St. Louis University; Philip Fung, Fenn College, Ohio; Ned Harrell, Menlo-Atherton High School, Atherton, California; J. A. H. Hunter, Toronto, Ontario, Canada; Richard A. Jacobson, South Dakota State University; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania; Sidney Spital, California State Polytechnic College; and the proposer.

An Ordering of the Rationals

571. [November, 1964] *Proposed by Herta Taussig Freitag, Hollins College, Virginia.*

According to Cantor's "diagonal procedure," the denumerability of the rationals may be established by ordering them in the manner indicated below:

$$\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdot & \cdot & \cdot \\[6pt]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdot & \cdot & \cdot \\[6pt]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdot & \cdot & \cdot \\[6pt]
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Thus,

$$\frac{1}{1} \leftrightarrow 1, \qquad \frac{1}{2} \leftrightarrow 2, \qquad \frac{2}{1} \leftrightarrow 3, \text{ etc.}$$

Design a matching formula between any given fraction a/b and the corresponding natural number n.

Solution by Henry W. Gould, West Virginia University.

There are two parts to this problem: (i) To show that to any given rational p/q we may assign a unique natural number n; (ii) To show that for a given natural number n we may exhibit a unique rational p/q. We give a constructive solution of both parts.

(i) Evidently a formula for n depends on the parity of $p+q$, this being so because of the alternating way in which the diagonals are traced out. We have in fact

$$n = \frac{(p+q-1)(p+q-2)}{2} + q, \quad \text{if } p+q \text{ is even}$$

and

$$n = \frac{(p+q-1)(p+q-2)}{2} + p, \quad \text{if } p+q \text{ is odd.}$$

These are based on nothing more complicated than the observation that n is given by counting in unit steps from one triangular number $(1, 3, 6, 10, 15, 21, \cdots)$ to the next.

(ii) To find a constructive way of actually exhibiting which rational p/q is assigned to a given natural number n we make use of the formula

$$(*) \qquad a = a_n = \left[\frac{1 + [\sqrt{(8n - 7)}]}{2}\right],$$

which, for natural numbers n, generates the curious sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \cdots,$$

each natural number k occurring precisely k times here. The formula in this form was called to the author's attention by some lecture notes of Leo Moser (Canadian Mathematical Congress lectures). In the formula, $[x]$ denotes the greatest integer $\leqq x$. Essentially the same formula occurs in E. S. Keeping's solution to Problem E1164 in the *American Mathematical Monthly* (1955, p. 731), another problem about finding which rational is assigned in a certain ordering. The solution of another *Monthly* problem, E1382, can also be made to depend on the same formula. An older reference of interest is to Problem 91, Page 271, in the 1939 edition (1953 reprint) of *Advanced Algebra* by S. Barnard and J. M. Child.

In any event, it is easy to determine from the formula that the sequence $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \cdots$ is generated by $n - \binom{a}{2}$, and the sequence is generated by

$$\binom{a+1}{2} - (n - 1).$$

These are the two sequences on which the pattern is based, and so we easily find that the particular p and q corresponding to a given value of n may be determined as follows:

We always have $p+q=a+1$, a being given by $(*)$. Then

$$\begin{aligned} n - \binom{a}{2} &= p, \quad \text{if } p + q \text{ is odd,} \\ &= q, \quad \text{if } p + q \text{ is even;} \\ \binom{a+1}{2} - (n - 1) &= q, \quad \text{if } p + q \text{ is odd,} \\ &= p, \quad \text{if } p + q \text{ is even.} \end{aligned}$$

An example will illustrate the ease of calculation. What is the 1000th rational assigned in the ordering? We have

$$p + q = a + 1 = 1 + \left[\frac{1 + [\sqrt{(7993)}]}{2}\right] = 46.$$

Then

$$p = \binom{46}{2} - 999 = 36 \quad \text{and} \quad q = 1000 - \binom{45}{2} = 10,$$

so that the 1000th rational is 36/10. This is easily checked by part (i); indeed $\frac{1}{2}(45)(44)+10=1000$.

Also solved by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; John A. Burslem, St. Louis University; Alan K. Chelgren, Centre College, Kentucky; Robert E. Cohen, University of Chicago; R. J. Cormier, Northern Illinois University; Ronald DeLaite, Orono, Maine; Monte Dernham, San Francisco, California; Robert V. Esperti, General Motors Defense Research Laboratories; Philip Fung, Fenn College, Ohio; Edwin V. Gadecki, Technical Operations Research, Burlington, Massachusetts; Harry W. Hickey, Arlington, Virginia; Roy H. Hines, Concord, Massachusetts; Stephen Hoffman, Trinity College, Connecticut; Richard A. Jacobson, South Dakota State University; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Herbert R. Leifer, Pittsburgh, Pennsylvania; Douglas Lind, University of Virginia; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania; Wade E. Philpott, Lima, Ohio; B. E. Rhoades, CUPM, Berkeley, California; Lawrence A. Ringenberg, Eastern Illinois University; J. R. Senft, DePaul University Chicago; Arnold Singer, Institute of Naval Studies, Cambridge, Massachusetts; Sidney Spital, California State Polytechnic College; Myron Tepper, University of Illinois; A. M. Vaidya, Texas Technological College; James A. Will, SUNY at Fredonia, New York; Charles Ziegenfus, Madison College, Virginia; and the proposer.

Rhoades found a solution in Zehna and Johnson, *Elements of Set Theory*, Boston, 1962, p. 108.

Space does not permit the publication of a number of interesting and different ways of matching p/q and n that were submitted.

Comment on Q343

Q343 [September, 1964] *Comment by Charles W. Trigg, San Diego, California.*

The published answer states "The angle θ evidently is 15°." It is further evident that θ is $15° \pm k \cdot 360°$, and for the right and left hand expressions, θ also is $135° \pm k \cdot 360°$.

Comment on Q347

Q347 [September, 1964] *Comment by Charles W. Trigg, San Diego, California.*

The method given shows

$$(1234)^2 = (1000)^2 + (200)^2 + (30)^2 + 4^2 + 2(1000)200 + 2(1200)30 + (1230)4.$$

This involves *thirteen* digit multiplications and the writing down of *seventeen* digits (exclusive of the final product):

$$\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 4 & & 9 & 1 & 6 \\
 & 4 & & & & & \\
 & & 7 & 2 & & & \\
 & & & 9 & 8 & 4 & \\
\hline
1 & 5 & 2 & 2 & 7 & 5 & 6
\end{array}$$

The zeros are necessary as place locaters.

It is interesting to compare this with the orthodox multiplication:

$$\begin{array}{ccccccc}
 & & & 4 & 9 & 3 & 6 \\
 & & 3 & 7 & 0 & 2 & \\
 & 2 & 4 & 6 & 8 & & \\
1 & 2 & 3 & 4 & & & \\
\hline
1 & 5 & 2 & 2 & 7 & 5 & 6
\end{array}$$

This involves *sixteen* digit multiplications and the writing down of *sixteen* digits in a drill-established pattern with built-in protection against misplacement of the partial products. Also, it has a built-in accuracy check, in that $1234+4936=2468+3702$.

Which really is quicker?

QUICKIES

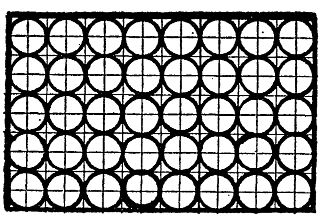
From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q359. Minimize $\int_0^1 F'(x)^2dx$ where $F(0)=0$ and $F(1)=1$.
[*Submitted by M. S. Klamkin*]

Q360. A cabinet maker uses the approximation that the diagonal of a square one foot on a side is 17 inches. A mathematician uses the approximation that $2-\sqrt{2}\doteq\sqrt{3}/3$. His son uses the approximation that $\sqrt{8}\doteq 2\frac{5}{6}$. Which approximation is more nearly correct?
[*Submitted by James H. Hill, Jr.*]

Q361. Find a geometric solution for the functional equation $F(2\theta)=F(\theta)\cos\theta/2$.
[*Submitted by M. S. Klamkin*]

Q362. Forty cylinders, each of one inch diameter, are placed inside a container measuring 8 inches by 5 inches as shown in the figure. Show how to place 41 of the one-inch cylinders into the same space.
[*Submitted by Sidney Kravitz*]



Q363. Factor $x^{11}+x^4+1$.
[*Submitted by M. S. Klamkin*]

(Answers on page 159)

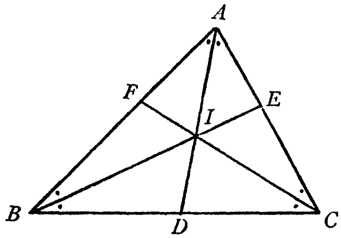


FIG. 2.

$$\text{In } \triangle ABD, \frac{AI}{ID} = \frac{AB}{BD}, \quad \text{and in} \quad \triangle ACD, \frac{AI}{ID} = \frac{AC}{CD}.$$

Hence

$$\frac{AI}{ID} = \frac{AB}{BD} = \frac{AC}{CD} = \frac{AB + AC}{BD + CD} = \frac{AB + AC}{BC} > 1.$$

Thus $AI > ID$. Similarly, we can show that $BI > IE$, $CI > IF$.

ANSWERS

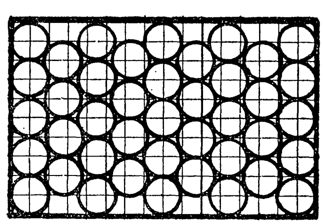
A359. By the Schwartz inequality

$$\int_0^1 F'(x)^2\, dx \int_0^1 dx \geqq \left\{ \int_0^1 F'(x)\, dx \right\}^2 = 1.$$

A360. All three of the approximations involve the proposition that $\sqrt{288}$ is approximately equal to $\sqrt{289}$.

A361. $F(\theta)$ denotes the distance the C. G. of a sector of angle 2θ is from the center. Consequently $F(\theta) = \sin \theta / 2\theta$.

A362. See the figure below.



A363. If w is a primitive cube root of unity, it follows immediately that $w^{3m+2} + w^{3n+1} = 0$. Consequently $x^2 + x + 1$ is a factor of $x^{3m+2} + x^{3n+1} + 1$. To find other factors, just divide.

(Quickies on page 188)